Higher Mathematics

Ya. S. BUGROV S. M. NIKOLSKY

differential and integral calculus

MIR PUBLISHERS MOSCOW

Higher Mathematics

ИЗДАТЕЛЬСТВО • МИР •

ВЫСШАЯ МАТЕМАТИКА

Я.С. Бугров
С.М. Никольский

ДИФФЕРЕНЦИАЛЬНОЕ И ИНТЕГРАЛЬНОЕ ИСЧИСЛЕНИЕ

Издательство «Наука» Москва

Higher Mathematics

Ya. S. BUGROV
S. M. NIKOLSKY

differential and integral calculus

Translated from the Russian
by
Leonid Levant

MIR PUBLISHERS
MOSCOW

First published 1982
Revised from the 1980 Russian edition

На английском языке

CONTENTS

PREFACE

This is the second book of our series "Higher Mathematics". The series consists of the following three books:

(1) *Fundamentals of Linear Algebra and Analytical Geometry*;

(2) *Differential and Integral Calculus*;

(3) *Differential Equations. Multiple Integrals. Series. Functions of a Complex Variable.*

The book presupposes no special preparation beyond the ability to perform ordinary algebraic operations and some elementary knowledge of geometry acquired at high school. Though the text (Chapters 1 to 7) centres on calculus of one variable, analytic geometry and calculus of several variables are also developed (in Chapter 8). The closing chapter (9) deals with series.

We think that this book and the first book of the series should be studied parallelly. Therefore, both books are provided with relevant cross-references.

Our series "Higher Mathematics" will be completed with a book of problems for all the three books. The latter will appear in 1983.

The Authors

CHAPTER 1

INTRODUCTION

Sec. 1.1. The Subject of Mathematics. Variable and Constant Quantities, Sets

Among all sciences mathematics occupies a special place. It is defined as the science studying space forms and quantitative relationships of real world. Of course, taking into consideration both the contemporary state of mathematics and variety of structures to be studied by it, space forms and quantitative relationships should be understood in the most general form.

Mathematics provides for other sciences the language of numbers and symbols suitable for expressing various relations between natural phenomena. But prior to applying mathematics, the biologist, physicist, or economist must attain a profound understanding of the essence of the phenomenon under study and partition it into parts subject to mathematical treatment.

Mathematics itself is concerned with the study of logical models designed to describe various phenomena of nature and society. Mathematics just studies the relationships between the elements of these models. If a mathematical model correctly reflects the essence of a given phenomenon, then it makes it possible to reveal some regularities hidden at the beginning of the study, i.e. mathematics is capable of revealing the qualitative aspect of a phenomenon.

By virtue of high abstractness, one and the same mathematical model can describe different processes. For instance, one and the same differential equation describes both the character of radioactive decay and a change in the temperature of a body.

In studying various phenomena of nature and society, we frequently come face to face with the variation of

quantities, with dependence of one of the quantities upon the other. Therefore, the concept of a variable quantity is a basic one in mathematical analysis.

By a *variable quantity* (or, simply, *variable*) we shall understand a quantity which, during the process of studying some phenomenon, attains at least two different values. A quantity which, during the investigation of a given problem, takes on only one value is called a *constant*.

Frederick Engels noted that the introduction of the Cartesian variable had introduced motion and dialectics into mathematics.

Uniting all the values attained by a variable, we obtain the *set* of values of this variable.

A *set* is also a basic concept in mathematics. This is a simple primary notion and we shall not try to define it via other simple notions.

A *set* is a collection of particular objects (things) of an arbitrary nature.

For instance, we may speak of the set of students of some institute, the set of molecules in a given body, the set of colour TV sets in a given hall, and so on. The objects belonging to a given set will be called *members* (or *elements*) of the set.

Sets are usually denoted by the capital letters A, B, . . ., X, Y, . . ., and their members by lower-case letters a, b, . . ., x, y, ...

If x is a member of the set A, or in other words, if x belongs to the set A, then we write: $x \in A$ (we read the membership symbol $\in$ as "is a member of, or belongs to"). If x is not a member of this set, then we write: $x \bar{\in} A$ (which is read as "x does not belong to A"). The symbol $A \subset B$ (meaning "the set A is included in the set B") denotes the fact that if $x \in A$, then $x \in B$.

In this case, the set A is called a *subset* of the set B. We also use the equivalent notation $B \supset A$ (meaning "the set B contains the set A"). The symbols $\subset$, $\supset$ are called *inclusion signs*.

If a set contains no elements, then it is called *empty* and is denoted by the symbol $\varnothing$. It is clear that $\varnothing \subset A$, where A is any set.

Sets are frequently denoted by braces inside which the elements constituting these sets are described in a certain way. The expression $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the *set of natural numbers*, the expression $\{0, 1, 2, \ldots\}$ the set of nonnegative integers, and the expression $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ the *set of all integers*. Here is one more example: $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\}$ is the set consisting of the digits of the decimal number system. Obviously, $2 \in A$, and $\frac{1}{2} \notin A$.

Sets A and B are considered to be *equal*, written $A = B$, if $A \subset B$ and $B \subset A$. The question concerning the equality of given sets is by far not always easy to be answered. For instance, if $A = \{6, 8, 10, \ldots\}$, and $B = \{p + q\}$ is the set of sums of the prime numbers p and q greater than 2, then it is clear that $B \subset A$. But it is not yet proved whether it is true that $A \subset B$, i. e. whether it is possible to represent any even number greater than, or equal to, 6 in the form of a sum of two prime numbers exceeding 2.

In the present book we shall mainly deal with numerical sets, whose elements are numbers.

Sec. 1.2. Operations with Sets

For a more comprehensive study of sets, it is advisable to introduce the arithmetic operations of addition and multiplication which possess a number of properties in many respects analogous to the corresponding properties of the operations of addition and multiplication on numbers.

Let there be given two arbitrary sets A and B. The *sum* or *union* of the sets A and B is defined as a set C consisting of the elements of the sets A and B. In this case we write $C = A + B$ or $C = A \cup B$ (Fig. 1). It is easily seen that $A + A = A$.

The set of all elements which belong both to A and B is called the *product* or *intersection* of the sets A and B, written AB or $A \cap B$ (Fig. 2). It is obvious that $A \cap A = A$. If $AB = \varnothing$, then we shall say that the sets A and B do not intersect. Utilizing the notion of the equa-

lity of sets, we can prove that: (1) $A + B = B + A$, (2) $(A + B)C = AC + BC$, (3) $(AB)C = A\,(BC)$, (4) $(A + B) + C = A + (B + C)$. For instance, let us prove (2). If $x \in (A + B)C$, then, according to the definition

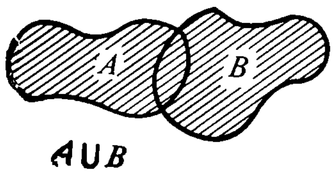

Fig. 1

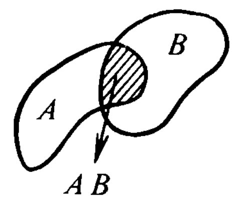

Fig. 2

of the product, $x \in A + B$ and $x \in C$. From the definition of the sum it follows that $x \in A$ or $x \in B$. Let, for the sake of definiteness, $x \in A$. Then $x \in AC$, and, consequently, $x \in AC + BC$. Hence, $(A + B)C \subset AC + BC$. If now

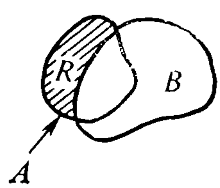

Fig. 3

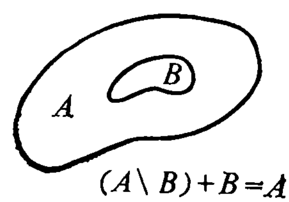

Fig. 4

the element $x \in AC + BC$, then at least one of the relations $x \in AC$, $x \in BC$ is fulfilled. For the sake of definiteness, let it be $x \in AC$. Then $x \in A + B$ and $x \in C$, i.e. $x \in (A + B)C$. Hence, $AC + BC \subset (A + B)C$, which proves equality (2).

The set consisting of all elements of A that do not belong to B is called the *difference* between the sets A and B and is denoted $R = A \setminus B$.

Note that in the general case $(A \setminus B) + B \neq A$ (Fig. 3). But if $B \subset A$, then $(A \setminus B) + B = A$ (Fig. 4).

Sets together with the above introduced operations of addition and multiplication form a distinctive algebra free of coefficients and powers.

Sec. 1.3. Symbols of Mathematical Logic

For the sake of getting briefer notations, we shall make use of some simple logical symbols. When interested not in the essence of some statement, but in its connection with other statements, we shall denote this statement by one of the letters α, β, ... The notation $\alpha \Rightarrow \beta$ means: "the statement β follows from the statement α", "if α, then β". The symbol $\alpha \Leftrightarrow \beta$ will be used to denote the fact that the statements α and β are equivalent, i.e. β follows from α and α follows from β.

The notation $\forall x \in A$: α means "for any element $x \in A$ the statement α is valid". The symbol $\forall$ is called a *universal quantifier.*

The notation $\exists y \in B$: β means "there is (exists) an element $y \in B$ for which the statement β holds". The symbol $\exists$ is termed an *existential quantifier.*

The symbol $\bar{\alpha}$ will be understood as the *negation* of the statement α or, briefly, "not α".

Let us form the negation of the statement $\forall x \in A$: α.

If the given statement does not take place, then the statement α is valid not for all $x \in A$, i.e. there exists an $x \in A$ for which α does not occur: $\overline{\forall x \in A:\ \alpha} \Leftrightarrow \exists x \in A:\ \bar{\alpha}$. Quite analogously, $\overline{\exists y \in B:\ \beta} \Leftrightarrow \forall y \in B:\ \bar{\beta}$.

Thus, to construct the negation of a given logical formula containing the symbols $\forall$ and $\exists$, it is necessary to interchange these symbols and to carry over the negation (line) onto the property standing after the colon. For instance, the negation of the statement

$$\exists M, \quad \forall x \in A: \quad f(x) \leqslant M$$

has the form

$$\overline{\exists M,\ \forall x \in A:\ \ f(x) \leqslant M} \Leftrightarrow \forall M,\ \ \exists x \in A:\ \ \overline{f(x) \leqslant M}$$
$$\Leftrightarrow \forall M,\ \exists x \in A:\ f(x) > M.$$

Sec. 1.4. Real Numbers

The number is a primary and basic concept in mathematics. This concept has traversed a long path of hystorical development. The set of natural numbers

$$\mathbf{N} = \{1, 2, 3, \ldots, n, \ldots\}$$

appeared due to the necessity to count various objects (hence, it is also called the set of counting numbers). Then, meeting the requirements of man's practice and the development of mathematics itself, there were introduced

integers (or whole numbers)

$$\mathsf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

and rational numbers

$$\mathsf{Q} = \{m/n\}, \text{ where } m, n \in \mathsf{Z},\ n \neq 0.$$

In order to get a single-valued notation of a rational number, we shall hold that the fraction m/n is irreducible, unless otherwise stated.

However, the introduction of rational numbers did not solve completely the important practical problem concerning the measurement of line segments. As a matter of fact, there are line segments whose lengths are not rational numbers. This can be exemplified by the diagonal of a square whose side is equal to unity.

This is why there arises the necessity of introducing numbers other than rational, viz. *irrational numbers*. Any rational or irrational number is called a *real* number. The set of real numbers is usually denoted by R. There exist various methods of introducing (defining) real numbers. We are going to dwell on one such method which represents them in the form of infinite nonterminating decimals:

$$a = \pm\, a_0.a_1\, a_2\, a_3 \ldots\ . \tag{1}$$

Here a_0 is some nonnegative integer, a_k are decimal digits. Thus, a_k can attain only one of the values 0, 1, 2, . . ., 9. The plus sign is frequently omitted in such notations.

The number 0 (zero) is written in one of the following forms:

$$0 = +\,0.00\ldots = 0.00\ldots = -\,0.00\ldots.$$

In order to represent a nonzero rational number $\pm m/n$ ($m > 0$, $n > 0$) in the form of a decimal, we carry out the process of division of m by n, using the known method learned at school:

$$\begin{array}{r|l} m & \quad n \\ \cline{2-2} & a_0 \cdot a_1 a_2 \ldots \end{array} \tag{2}$$

Note, that applying this method to the other notation of the fraction $\pm mp/np = \pm m/n$ $(p > 0)$, we get the same result.

We set

$$\pm \frac{m}{n} = \pm a_0.a_1a_2a_3 \ldots \qquad (3)$$

and call the right-hand side of (3) a *decimal expansion of the number* $\pm m/n$.

If the denominator of the fraction has the form $n = 2^s5^l$, where s, l are nonnegative integers, then process (2) comes to an end after a finite number of steps yielding a *terminating decimal*:

$$\pm \frac{m}{n} = \pm a_0.a_1 \ldots a_M$$
$$= \pm a_0.a_1 \ldots a_M00 \ldots \quad (a_M > 0). \qquad (4)$$

Another representation of terminating decimal (4) is also used:

$$\pm a_0.a_1 \ldots a_M = \pm a_0.a_1 \ldots a_{M-1}(a_M - 1)\,99\ldots$$
$$= \pm a_0.a_1 \ldots a_{M-1}(a_M - 1)\,(9), \qquad (5)$$

though it does not follow from process (2). The representation on the right in (5) is a *nonterminating decimal.*

The decimal

$$\beta_0.\beta_1\beta_2 \ldots$$

is called *nonterminating* if for any natural n we can find a natural $k > n$ such that the digit β_k standing in the kth place after the decimal point is greater than zero.

We should like to underline that any decimal having the form $\pm a_0.a_1 \ldots a_M00 \ldots$ is called *terminating.* Taking advantage of method (5), any terminating decimal can be represented in the form of a nonterminating decimal. Other decimals to be treated below will be determined by other numbers.

Let now the denominator of our fraction do not have the form 2^s5^l. Then (2) is an endless process, since a positive remainder occurs at any step. Each remainder is less than n and, therefore (after all the digits of the number m

are used) at least two remainders, equal to each other, will occur among the first n remainders. But as soon as any remainder occurs a second time, the process becomes repeating or *periodic*, since the terms after this remainder will repeat also. Therefore, the decimal expansion of an arbitrary rational number has the form

$$\pm\frac{m}{n}=\pm a_0.a_1\ldots a_M b_1\ldots b_s b_1\ldots b_s\ldots$$
$$=\pm a_0.a_1\ldots a_M(b_1\ldots b_s)\quad (s<n). \qquad (6)$$

Expansion (5) may be regarded as a particular case of (6).

Examples.

$$\left.\begin{aligned}&\frac{1}{6}=0.166\ldots=0.1(6),\quad \frac{1}{7}=0.(142857),\\&\frac{2}{9}=0.22\ldots=0.(2),\\&\frac{7}{99}=0.0707\ldots=0.(07),\\&\frac{7}{990}=0.00707\ldots=0.0(07).\end{aligned}\right\}\qquad(7)$$

The expansion of form (6) is called an *infinite repeating decimal.*

Hence, any nonzero rational number can be expanded to an infinite repeating decimal with the aid of process (2), and in case (4) also by means of process (5). And it is possible to prove that to different rational numbers there correspond different nonterminating decimal expansions. The converse is also true: any infinite repeating decimal (6) is generated with the aid of processes (2) and (5), by some rational number computed by the formula

$$\pm\frac{m}{n}=\pm\left[a_0.a_1\ldots a_M+\frac{\beta_1\ldots\beta_s}{9\ldots9}10^{-M}\right].$$

Here we have denoted by $\beta_1\ldots\beta_s$ and $\underbrace{9\ldots9}_{s\text{ times}}$ a whole number written respectively by the digits $\beta_1,\ldots,\beta_s$ and $\underbrace{9,\ldots,9}_{s\text{ times}}$.

For instance,

$$1.237\,(06) = 1.237 + 0.000\,(06) = 1.237 + \frac{06}{99}\,10^{-3}$$

$$= 1.237 + \frac{6}{99\,000}.$$

Along with repeating decimals, there exist nonrepeating ones, for instance, 0.1010010001 . . .; 0.121122111222 . . . .

Here is another example: if a square root is taken out of 2 according to the known rule, then a definite infinite nonrepeating decimal is obtained: $\sqrt{2} = 1.41\ \ldots$. It is defined in the sense that to any natural number k there corresponds a definite digit α_k standing in the kth place after the decimal point and uniquely computable according to the rule for extracting a square root.

Mathematical analysis offers many ways of computing the number π with any preassigned accuracy. This results in a quite definite nonterminating decimal expansion of π which, as it turns out, is not a mixed repeating decimal.

Let us now give the definition of the irrational number, purely formal for the time being. *The irrational number is defined as an arbitrary infinite nonrepeating decimal*

$$a = \pm\,\alpha_0.\alpha_1\alpha_2\alpha_3\ \ldots, \tag{8}$$

where α_0 *is a nonnegative integer,* α_k $(k = 1, 2, \ldots)$ *are digits, and the equality sign* " = " *expresses the fact that we have denoted the right-hand member of* (8) *by* a. However, it is convenient to say that the right-hand member of (8) is a decimal expansion of the number a.

Rational and irrational numbers are called real numbers.

It follows from the aforesaid that *any nonzero real number can be written in the form of a nonterminating decimal* (8). *If it is rational, then its decimal expansion is an infinite repeating decimal. Otherwise, according to our definition, expression* (8) *itself defines an irrational number.*

A *nonzero decimal* can be terminating, but it does not define a new rational number: by virtue of the agreement expressed by equality (5), it can be replaced by an equal infinite repeating decimal.

The number a, *where not all* α_k *are zero, is said to be positive or negative* depending on whether the right-hand

member of (8) is preceded by "+" or "—", the plus sign being, as usual, omitted.

Thus, we have defined real numbers in a formal way. We still have to define the arithmetic operations on them, to introduce the notion "greater than" for them, and to verify that these operations and the notion "greater than" conform with the corresponding operations and the notion "greater than" already defined for rational numbers, as well as satisfy the properties possessed by numbers.

Sec. 1.5. Defining an Equality and Inequality

Let there be given two numbers $a = \pm \alpha_0.\alpha_1\alpha_2\ldots$, $b = \pm \beta_0.\beta_1\beta_2\ldots$, defined by *nonterminating* decimals. We shall hold that they are *equal* to each other if and only if they are of the same sign and

$$\alpha_k = \beta_k \quad (k = 0, 1, 2, \ldots).$$

Let a and b be positive numbers. Then $a < b$ or, which is the same, $b > a$ if $\alpha_0 < \beta_0$ or if there can be found an l (a nonnegative integer) such that $\alpha_k = \beta_k$ $(k = 0, 1, \ldots, l)$ and $\alpha_{l+1} < \beta_{l+1}$.

If nonterminating decimals a and b have the form

$$\left.\begin{aligned} a &= \alpha_0.\alpha_1\ldots\alpha_N 99\ldots \quad (\alpha_N < 9),\\ b &= \beta_0.\beta_1\ldots\beta_{N_1} 99\ldots \quad (\beta_{N_1} < 9), \end{aligned}\right\} \tag{1}$$

then they can be written in the form of terminating decimals

$$\left.\begin{aligned} a &= \alpha_0.\alpha_1\ldots\alpha_{N-1}(\alpha_N + 1)\\ b &= \beta_0.\beta_1\ldots\beta_{N_1-1}(\beta_{N_1} + 1). \end{aligned}\right\} \tag{1'}$$

It is easy to see that if $a = b$ or $a < b$ in the sense of the above given definition (in terms of infinite decimals (1)), then $a = b$ or respectively $a < b$ in the sense of the equality $(a = b)$ or inequality $(a < b)$ of terminating decimals (1′).

We define $a > 0$ or $a < 0$ depending on whether a is positive or negative; further, a is defined to be less than b if $a < 0$, $b > 0$, or if $a, b < 0$ and $|a| > |b|$.

If $a = \mp \alpha_0.\alpha_1\alpha_2 \ldots$, then, by definition, $-a = \mp\alpha_0.\ \alpha_1\alpha_2 \ldots$, and the absolute value $|a| = + \alpha_0.\alpha_1\alpha_2 \ldots = \alpha_0.\alpha_1\alpha_2 \ldots$. Hence,

$$|-a| = |a| = \begin{cases} a & (a \geqslant 0), \\ -a & (a \leqslant 0). \end{cases}$$

As we know, between real numbers and points of a certain straight line we can establish one-to-one correspondence ($\leftrightarrow$) according to the following rule. The number 0 is associated with an arbitrary point O on the line called

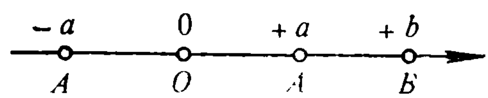

Fig. 5

the zero point, and vice versa. The length of some line segment is taken for unity. Every real number $\pm a$ $(a > 0)$ is associated with a point of the straight line situated at a distance a from the zero point; on the right of the point O for the number $+a$ and on the left of the point O for the number $-a$ (Fig. 5). Conversely, if A is an arbitrary point of the straight line found at a distance a on the right of the point O, then it is considered to correspond to the real number $+a$ (infinite decimal). And if the point A is located on the left of the point O, then it corresponds to the number $-a$. The straight line under consideration will be called the *number line* or the real axis. Below, we shall identify the points of the number line with the real numbers which correspond to them, i.e. the points themselves will be called by the corresponding numbers. Note that the distance between the points a and b is equal to $|a - b|$ (for the definition of difference see the next section).

Sec. 1.6. Defining Arithmetic Operations

Let, by virtue of a certain law, every nonnegative integer (*subscript*) n is associated with the number x_n. Then, the set

$$x_0,\ x_1,\ x_2,\ \ldots \tag{1}$$

is called a *sequence* (of numbers). Individual numbers x_n of sequence (1) are called its *terms*. The terms x_n and x_m for $m \neq n$ are considered to be different as terms of the sequence, although it is not excluded that they are equal to each other as numbers ($x_n = x_m$).

A *sequence* is termed *nondecreasing* (*nonincreasing*) if $x_k \leqslant x_{k+1}$ ($x_k \geqslant x_{k+1}$) for all $k = 0, 1, 2, \ldots$.

We shall say that *sequence* (1) *is bounded above* (*by a number* M) if there is a number M such that $x_k \leqslant M$ for all $k = 0, 1, 2, \ldots$.

If x_k in sequence (1) are integers, then we shall say that it is *stabilized* to the number ξ if there can be found k_0 such that $x_k = \xi$ for all $k > k_0$, and write $x_k \rightrightarrows \xi$.

Lemma 1. *If a sequence of integers does not decrease and is bounded above by a number M, then it is stabilized to a certain integer $\xi \leqslant M$.*

Proof. The sequence $\{x_n\}$ is bounded above by the number M and does not decrease, hence it is also bounded below. Then, although our sequence has an infinite number of terms, it runs through a *finite* number of integers. As a matter of fact, these numbers do not exceed M. Let ξ be the greatest among these numbers. Thus, $\xi \leqslant M$ and there exists such a natural s for which $x_s = \xi$. But our sequence does not decrease and therefore $x_k = x_s$ for all $k \geqslant s$, i.e. our sequence is stabilized to the number ξ ($x_n \rightrightarrows \xi \leqslant M$).

Let us now consider a sequence of nonnegative decimals (only terminating or only nonterminating):

$$\left.\begin{array}{l} a_1 = \alpha_{10}.\alpha_{11}\alpha_{12}\alpha_{13}\ \ldots, \\ a_2 = \alpha_{20}.\alpha_{21}\alpha_{22}\alpha_{23}\ \ldots, \\ a_3 = \alpha_{30}.\alpha_{31}\alpha_{32}\alpha_{33}\ \ldots, \\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\} \qquad (2)$$

The right-hand members in (2) form an array (*a nonterminating matrix*).

We shall say that *sequence* (2) *is stabilized to the number* $a = \gamma_0.\gamma_1\gamma_2\ldots$, and write

$$a_n \rightrightarrows a, \qquad (3)$$

if the k*th column of array* (2) *is stabilized to* γ_k, *whatever* $k = 0, 1, 2, \ldots$, *may be*, i.e. $\alpha_{sk} \rightrightarrows \gamma_k$ for any fixed k.

Lemma 2. *If a nondecreasing sequence* (2) *of decimals (only terminating or only nonterminating) is bounded above by a number* M, *then it is automatically stabilized to a certain number* a *which satisfies the inequalities*

$$a_n \leqslant a \leqslant M \quad (n = 1, 2, \ldots). \tag{4}$$

Indeed, under the conditions of the lemma the integers of the zero column of matrix (2)

$$\begin{array}{c} \alpha_{10} \\ \alpha_{20} \\ \alpha_{30} \\ \cdots \end{array}$$

do not decrease as well, and are bounded above by the number M (see the preceding section) and, therefore, according to Lemma 1, they are stabilized to some nonnegative integer $\gamma_0 \leqslant M$. Let this stabilization take place beginning with the number n_0, i.e. $a_n = \gamma_0.\alpha_{n1}\alpha_{n2} \ldots \ldots \leqslant M$, $n \geqslant n_0$.

Let us now prove that the first column in (2)

$$\begin{array}{c} \alpha_{11} \\ \alpha_{21} \\ \alpha_{31} \\ \cdots \end{array}$$

is also stabilized to a certain digit γ_1 and the following inequality is valid:

$$\gamma_0.\gamma_1 \leqslant M.$$

Indeed, since the decimal expansions of the numbers a_n for $n \geqslant n_0$ have the form

$$\gamma_0.\alpha_{n1}\alpha_{n2}\alpha_{n3}\ldots \leqslant M \quad (n \geqslant n_0)$$

and, besides, a_n does not decrease, for the indicated n's the digits of the first column α_{n1} ($\leqslant 9$) do not decrease either and, consequently, by Lemma 1, are stabilized to some digit γ_1. Let this second stabilization occur begin-

ning with the number $n_1 > n_0$, i.e. for $n \geqslant n_1$

$$a_n = \gamma_0 . \gamma_1 \alpha_{n2} \alpha_{n3} . . . \leqslant M$$

It is obvious here that

$$\gamma_0 . \gamma_1 \leqslant a_n \leqslant M \quad (n \geqslant n_1).$$

Using now the method of induction, let us assume that it was already proved that the columns of matrix (2) with the numbers not exceeding k are stabilized respectively to $\gamma_0, \gamma_1, \ldots, \gamma_k$ and

$$\gamma_0, \gamma_1 . . . \gamma_k \leqslant M \quad (\gamma_1, \ldots, \gamma_k \text{ digits}). \tag{5}$$

Let us prove that the $(k+1)$ st column in (2) is also stabilized to a certain digit γ_{k+1} and the following inequality is valid:

$$\gamma_0 . \gamma_1 . . . \gamma_k \gamma_{k+1} \leqslant M. \tag{6}$$

Indeed, since the decimal expansions of the numbers a_n for $n \geqslant n_k$ have the form

$$a_n = \gamma_0 . \gamma_1 . . . \gamma_k \alpha_{n,\, k+1} \alpha_{n,\, k+2} . . . \leqslant M$$

and, in addition, a_n does not decrease, then for the indicated n's the digits $\alpha_{n,\, k+1}$ $(\leqslant 9)$ do not decrease and, consequently, are stabilized for $n \geqslant n_{k+1}$ where n_{k+1} is sufficiently large, to a certain digit γ_{k+1}. It is obvious that

$$\gamma_0 . \gamma_1 . . . \gamma_{k+1} \leqslant a_n \leqslant M \quad (n \geqslant n_{k+1}),$$

which proves inequality (6). By virtue of the method of mathematical induction, we conclude that (5) is true for any k and that $a_n \rightrightarrows a = \gamma_0 . \gamma_1 \gamma_2 . . .$

Let us prove the first inequality of (4). Compare the numbers

$$a_n = \alpha_{n0} . \alpha_{n1} \, \alpha_{n2} \, \alpha_{n3} \ldots,$$
$$a = \gamma_0 . \gamma_1 \, \gamma_2 \, \gamma_3 \ldots.$$

If all the corresponding components of both expansions are equal to each other ($\alpha_{ns} = \gamma_s$, $s = 0, 1, 2, \ldots$), then $a_n = a$. Otherwise, for a certain s

$$\left.\begin{array}{ll} \alpha_{nj} = \gamma_j & (j = 0, 1, \ldots, s-1), \\ \alpha_{ns} < \gamma_s & \end{array}\right\} \tag{7}$$

And if $s = 0$, then the equality in (7) should be omitted. For the n under consideration the numbers $\alpha_{nj} = \gamma_j$ $(j = 0, 1, \ldots, s-1)$

are already stabilized, therefore

$$a_n \leqslant \alpha_{n0}.\alpha_{n1} \ldots \alpha_{n,\,s-1}(\alpha_{ns}+1) = \gamma_0.\gamma_1 \ldots \gamma_{s-1}(\alpha_{ns}+1)$$
$$\leqslant \gamma_0.\gamma_1 \ldots \gamma_{s-1}\gamma_s \leqslant a.$$

Thus, we have proved the first inequality of (4).

Let us now prove the second inequality of (4). If $a = \gamma_0.\gamma_1 \ldots \gamma_N$ is a terminating decimal, then this inequality follows from (5) for $k = N$. Let now

$$a = \gamma_0.\gamma_1\gamma_2 \ldots \tag{8}$$

be a nonterminating decimal. We expand the number M to obtain also a nonterminating decimal $M = m_0.m_1m_2 \ldots$. If we assume that the inequality being proved is false, then we can find a k such that

$$\left.\begin{array}{ll} \gamma_j = m_j & (j=0,\ 1,\ \ldots,\ k-1), \\ \gamma_k > m_k. & \end{array}\right\} \tag{9}$$

If $k = 0$, then the equalities in (9) are omitted. Since expansion (8) is nonterminating, we can find an s such that $\gamma_{k+s} > 0$. Therefore

$$\gamma_0.\gamma_1 \ldots \gamma_{k-1}\gamma_k \ldots \gamma_{k+s} > \gamma_0.\gamma_1 \ldots \gamma_{k-1}\gamma_k = m_0.m_1 \ldots m_{k-1}\gamma_k$$
$$\geqslant m_0.m_1 \ldots m_{k-1}(m_k+1) = m_0.m_1 \ldots m_{k-1}m_k99 \ldots$$
$$\geqslant m_0.m_1m_2 \ldots = M,$$

and we have arrived at a contradiction with inequality (5).

Now it is possible to define the arithmetic operation with real numbers.

For an arbitrary number $a = \alpha_0.\alpha_1\alpha_2 \ldots$ let us introduce its nth *truncation* $a^{(n)} = \alpha_0.\alpha_1 \ldots \alpha_n$ which is a terminating decimal. We are sure that the reader is familiar with the operations with terminating decimals.

Let there be given two positive numbers

$$a = \alpha_0.\alpha_1\alpha_2 \ldots, \qquad b = \beta_0.\beta_1\beta_2 \ldots,$$

expanded into nonterminating decimals. Let us introduce the following sequence of numbers

$$a^{(n)} + b^{(n)} = \alpha_0.\alpha_1 \ldots \alpha_n + \beta_0.\beta_1 \ldots \beta_n$$
$$= \lambda_0^{(n)}.\lambda_1^{n} \ldots \lambda_n^{(n)}$$
$$(n = 1,\ 2,\ \ldots).$$

Obviously, this sequence does not decrease, besides, it is bounded above

$$a^{(n)} + b^{(n)} \leqslant (\alpha_0 + 1) + (\beta_0 + 1) \quad (n = 1,\ 2,\ \ldots),$$

But then, by Lemma 2, the decimal expansions of our sequence are stabilized to a certain decimal $\gamma_0.\gamma_1\gamma_2...$, which is a real number. This number is just defined as the *sum of the numbers* a and b:

$$a + b = \gamma_0.\gamma_1\gamma_2 \dots$$

Thus, we define the sum $a + b$ as the number to which $a^{(n)} + b^{(n)}$ is stabilized

$$a^{(n)} + b^{(n)} \rightrightarrows a + b. \tag{10}$$

In order to define the product of two positive numbers a and b, let us introduce the following truncated number:

$$(a^{(n)}b^{(n)})^{(n)} = \mu_0^{(n)} \cdot \mu_1^{(n)} \dots \mu_n^{(n)} \tag{11}$$

which is a terminating decimal. The sequence of these truncations, obviously, does not decrease (with an increase in n!) and is bounded above:

$$(a^{(n)}b^{(n)})^{(n)} \leqslant (\alpha_0 + 1)(\beta_0 + 1) \quad (n = 1, 2, \dots).$$

Therefore, by Lemma 2, expression (11) is stabilized to a certain number which is just called the *product* ab:

$$(a^{(n)}b^{(n)})^{(n)} \rightrightarrows ab.$$

Let us turn our attention to the following inequalities:

$$a^{(n)} = \alpha_0.\alpha_1 \dots \alpha_n \leqslant \alpha_0.\alpha_1 \dots \alpha_n \dots \leqslant \alpha_0.\alpha_1 \dots \alpha_n 99 \dots$$
$$= \alpha_0.\alpha_1 \dots \alpha_{n-1}(\alpha_n + 1) = \alpha_0.\alpha_1 \dots \alpha_n + 10^{-n},$$

i.e.

$$a^{(n)} \leqslant a \leqslant a^{(n)} + 10^{-n}$$

The quantity $a^{(n)}$ approaches a (as n increases) without decreasing. As far as the quantity $a^{(n)} + 10^{-n}$ is concerned, it approaches a without increasing:

$$a^{(n)} + 10^{-n} = \alpha_0.\alpha_1 \dots \alpha_n 99 \dots \geqslant \alpha_0.\alpha_1 \dots \alpha_n\alpha_{n+1}99 \dots$$
$$= a^{(n+1)} + 10^{-(n+1)}.$$

This point will be referred to when defining the difference and the quotient of positive numbers.

If $a > b > 0$, then the *difference* $a - b$ is defined as a decimal (number) to which the sequence of terminating

decimals is stabilized:

$$a^{(n)} - (b^{(n)} + 10^{-n}) \rightrightarrows (a - b); \tag{12}$$

if a, $b > 0$, then the *quotient* a/b is defined as a decimal to which the sequence of terminating decimals is stabilized:

$$\left(\frac{a^{(n)}}{b^{(n)}+10^{-n}}\right)^{(n)} \rightrightarrows \frac{a}{b}. \tag{13}$$

It should be taken into consideration that, with an increase in n, $a^{(n)}$ does not decrease, and $b^{(n)} + 10^{-n}$ does not increase, and therefore the expressions on the left of (12) and (13) do not decrease. Besides, they are bounded above:

$$a^{(n)} - (b^{(n)} + 10^{-n}) \leqslant \alpha_0 + 1,$$

$$\left(\frac{a^{(n)}}{b^{(n)}+10^{-n}}\right)^{(n)} \leqslant \frac{\alpha_0+1}{\beta_0 \cdot \beta_1 \ldots \beta_s},$$

where s is such that $\beta_s > 0$. Therefore, by Lemma 2, the expressions on the left in (12) and (13) are really stabilized.

Let us also set

$$0 + a = a \pm 0 = a, \quad a \cdot 0 = 0 = a - a$$
$$= \frac{0}{b} \quad (a \geqslant 0,\ b > 0). \tag{14}$$

We have defined for nonnegative numbers a and b their sum, difference, product, and quotient, assuming that $a \geqslant b$ in the case of difference and that $b > 0$ in the case of quotient. These definitions are extended by ordinary methods to the numbers a and b of arbitrary signs. For instance, if a, $b \leqslant 0$, then we set $a + b = b + a = -(|a| + |b|)$. And if a and b are numbers of opposite signs and $|a| \geqslant |b|$, then we set $a + b = b + a = \pm(|a| - |b|)$, where the sign is chosen the same as one preceding a. In particular,

$$a + (-a) = 0$$

is valid for any a.

Similar rules could be given for the remaining arithmetic operations, but we are not going to formulate them, since they are well known from the course of algebra.

The next section deals with the properties of real numbers that follow from the above given definitions. We only formulate these properties. The reader is advised to accept them on faith*. These properties are subdivided into five groups (I to V). The first three of them comprise the elementary properties we are guided in carrying out arithmetic calculations and solving inequalities. Group IV reflects one property (Archimedean). Finally, Group V also consists of one property. This property is formulated in terms of limits. It will be proved later on, in Sec. 2.5.

Sec. 1.7. Basic Properties of Real Numbers

I. Order Properties.

I_1. For every pair of real numbers a and b there holds one and only one of the relations

$$a = b, \quad a > b, \quad a < b.$$

I_2. $a < b$ and $b < c$ imply $a < c$ (the transitivity of the relation expressed by the "less than" sign).

I_3. If $a < b$, then there exists a number c such that $a < c < b$.

II. Properties of the Operations of Addition and Subtraction.

II_1. $a + b = b + a$ (commutativity).

II_2. $(a + b) + c = a + (b + c)$ (associativity).

II_3. $a + 0 = a$.

II_4. $a + (-a) = 0$.

II_5. $a < b$ implies that $a + c < b + c$ for any c.

The number $a + (-b)$ should be naturally called the difference $a - b$, i.e. written $a - b = a + (-b)$, since if it is added to b, we get a:

$$[a + (-b)] + b = a + [(-b) + b] = a + 0 = a.$$

It easily follows from the above properties that the difference is unique. We can prove that the difference thus

* In Sec. 1.7 we give the proof only for some particular cases. For a complete proof see, for instance, *A Course of Mathematical Analysis* (a textbook for university students) by S.M. Nikolsky, vol. 1, Ch. 2, Mir Publishers, Moscow, 1977.

defined coincides with the difference defined by formula (12) of the preceding section.

III. Properties of the Operations of Multiplication and Division.

III_1. $ab = ba$ (commutativity).

III_2. $(ab)\,c = a\,(bc)$ (associativity).

III_3. $a \cdot 1 = a$.

III_4. $a \cdot \frac{1}{a} = 1$ $(a \neq 0)$.

III_5. $(a + b)\,c = ac + bc$ (distributivity).

III_6. $a < b$, $c > 0$ implies $ac < bc$.

The number $a \cdot \frac{1}{b}$ $(b \neq 0)$ should be naturally called the quotient $\frac{a}{b}$ $\left(\frac{a}{b} = a \cdot \frac{1}{b}\right)$, since, when multiplied by b, it yields a:

$$\left(a \cdot \frac{1}{b}\right) b = a \left(\frac{1}{b} \cdot b\right) = a \left(b \cdot \frac{1}{b}\right) = a \cdot 1 = a.$$

It follows easily from the aforegoing properties that the quotient $\frac{a}{b}$ is unique. It is possible to prove that the quotient thus defined coincides with the quotient defined by formula (13) from the preceding section.

IV. Archimedean Property.

Given any number $c > 0$, there exists a natural number $n > c$. Indeed, if $c = \alpha_0.\alpha_1\alpha_2\ldots$, then we can put $n = \alpha_0 + 2$.

The Archimedean property and some of the previous properties imply that, given any positive number ε, there always is a natural number n such that the inequality $1/n < \varepsilon$ is fulfilled.

Indeed, according to IV, for the number $1/\varepsilon$ there exists a natural number n such that $1/\varepsilon < n$, which, by III_6, implies the required inequality.

Note that for every given number $c \geqslant 0$ there is, obviously, a single number m in the sequence $0, 1, 2, \ldots$ of nonnegative integers for which the inequalities $m \leqslant c < < m + 1$ hold.

Property V. *If a sequence of real numbers* $a_1, a_2, a_3, \ldots$ *does not decrease and is bounded above by a number* M $(a_n \leqslant \leqslant M)$, *then there exists a number* $a \leqslant M$ *to which this se-*

quence tends as to its limit:

$$\lim_{n\to\infty} a_n = a \leqslant M.$$

This means that for any arbitrarily small positive number $\varepsilon > 0$ we can find a natural number n_0 such that

$$|a - a_n| = a - a_n < \varepsilon$$

for all $n > n_0$.

For the proof of this property see Sec. 2.5 (Theorem 1). As it will be shown, Property V is an immediate consequence of Lemma 2 from the preceding section. As we recall, this lemma stated that a nondecreasing sequence of nonterminating decimals bounded above by a number M is stabilized to a certain decimal $a \leqslant M$ $(a_n \rightrightarrows a)$.

The point is that the stabilization of a_n to a implies that a_n tends to a as to its limit.

Sec. 1.8. Axiomatic Approach to the Theory of Real Numbers

We called nonterminating decimals real numbers, introduced for them the notions 0, 1, $>$, $=$, arithmetic operations and formulated their basic properties (I-V) which can be proved.

It should be noted that properties I to V are chosen so economically and comprehensively that from them we can logically derive all other properties of numbers.

There exists an axiomatic approach to the definition of real numbers consisting in that real numbers are defined as some objects (things) $a, b, c, \ldots$, which satisfy properties I-V. In the axiomatic approach these properties are called *axioms of number.*

When treated axiomatically, the formulations of the mentioned properties (now axioms) must be somewhat modified. Now axioms II are formulated as follows: to each pair of numbers a, b there should correspond, in accordance with a certain law, a number $a + b$ called their sum, and axioms II_1-II_5 are fulfilled. Axiom II_3 should then be modified to read: there exists number 0 (zero) such that $a + 0 = a$ for all a's. Axiom II_4 is now formulated in the following way: for any number a there

exists a number, denoted by $-a$, such that $a + (-a) = 0$. Finally, axiom III_3 takes the form: there exists a number 1 (unity) different from zero and such that $a \cdot 1 = a$ for all a.

Let us denote by R the set of all real numbers, i.e. of all things obeying axioms I-V. Then R contains both zero 0 and unity 1. With the aid of the axioms, we can prove that $0 < 1$ and the numbers $2 = 1 + 1$, $3 = 2 + 1$, . . . and the numbers -1, -2, -3, . . . have sense. As a result, we shall obtain the set of all integers (not equal to one another!)

$$\ldots, -2, -1, 0, 1, 2, \ldots .$$

By virtue of the above axioms, these numbers may be divided by one another, except for the division by 0. Therefore, contained in R are rational numbers $\pm m/n = \pm mp/np$ $(n > 0,\ m \geqslant 0,\ p \neq 0)$. But then R also contains terminating decimals which can form nondecreasing sequences bounded above. By axiom V, R must also contain the limits of such sequences. Some of these limits are not terminating decimals, these are numbers different from terminating decimals. It is convenient to write them in the form of nonterminating decimals. As a result of logical reasoning, from the axioms we have come to nonterminating decimals. Of course, we have presented here only a scheme for logical reasoning which does not pretend to be a proof.

From the aforesaid it follows that from the formal viewpoint it is all the same whether, in defining real numbers, we proceed from nonterminating decimals or from the axiomatic approach.

Of course, from philosophical point of view the second approach is more acceptable: numbers are abstractions expressing quantitative relationships of the real world, while decimals are formal symbols representing them.

Sec. 1.9. Inequalities for Absolute Values

The inequality

$$|a| < \varepsilon \tag{1}$$

is equivalent to the two inequalities

$$-\varepsilon < a < \varepsilon. \tag{1'}$$

Hence the inequality

$$| a - b | < \varepsilon \tag{2}$$

is equivalent to the inequalities

$$b - \varepsilon < a < b + \varepsilon. \tag{2'}$$

Analogously, the inequality

$$| a - b | \leqslant \varepsilon \tag{3}$$

is equivalent to the inequalities

$$b - \varepsilon \leqslant a \leqslant b + \varepsilon.$$

There also hold the inequalities

$$| a + b | \leqslant | a | + | b |, \tag{4}$$

and

$$| a - b | \geqslant || a | - | b ||. \tag{5}$$

Inequality (4) can be derived by considering separately the four possible cases: (1) $a, b \geqslant 0$, (2) $a, b \leqslant 0$, (3) $a \leqslant 0 \leqslant b$, and (4) $b \leqslant 0 \leqslant a$.

For instance, in case (2) we have

$$a + b \leqslant b \leqslant 0 \quad \text{and} \quad | a + b | = -(a + b)$$

$$= -a - b = | a | + | b |,$$

while in case (3), assuming that $| b | \geqslant | a |$, we obtain

$$| a + b | = b + a \leqslant | a | + | b |.$$

Let the reader consider case (3) by analogy with case (1) under the consumption $| b | \leqslant | a |$, case (4) is reducible to case (3).

Further, by (4), we have

$$| a | \leqslant | b | + | a - b |, \; | b | \leqslant | a | + | a - b |,$$

i.e.

$$| a | - | a - b | \leqslant | b | \leqslant | a | + | a - b |,$$

whence follows (5).

Sec. 1.10. Intervals, Bounded Sets

Let some numbers (points) a and b satisfy the inequality $a < b$.

The set of all numbers x satisfying the inequalities $a \leqslant x \leqslant b$, is referred to as a *closed interval* (with end points a and b) and is denoted as $[a, b]$.

The set of the numbers x satisfying the inequalities $a < x < b$ is spoken of as an *open interval* (with end points a and b) and is denoted (a, b).

The set of the numbers x satisfying the inequalities $a \leqslant x < b$ or $a < x \leqslant b$, is denoted, respectively, as $[a, b)$, or $(a, b]$ and is called a *half-open* (semi-open) or *half-closed interval* (also, simply, a *half-interval*).

We also frequently consider sets referred to as *infinite intervals* or *half-intervals* of the following types: (1) $(-\infty, \infty)$, (2) $(-\infty, a]$, (3) $(-\infty, a)$, (4) (a, ∞), and (5) $[a, \infty)$.

The first of them is the set of all real numbers (of all points of the real line), while the others and respectively the sets containing the numbers x satisfying the inequalities: (2) $x \leqslant a$, (3) $x < a$, (4) $a < x$, (5) $a \leqslant x$.

For the sake of convenience, the symbols $-\infty$ and $+\infty$ will be called *infinite numbers*, and ordinary numbers — *finite numbers*.

Note that the end points of the closed interval $[a, b]$ are finite numbers, whereas the "end points" of the open interval (a, b) may be both finite and infinite. The number a of the half-interval $[a, b)$ is always finite, while the number b may be either finite or infinite ($b \leqslant \infty$). Analogously, the number a of the half-interval $(a, b]$ is either finite or infinite ($-\infty \leqslant a$), b being always finite.

If a and b are finite and $a < b$, then the number $b - a$ is called the *length* of the closed interval $[a, b]$ or of the open interval (a, b), or of the half-intervals $(a, b]$, $[a, b)$.

An arbitrary interval (a, b) containing the point c ($a < c < b$) will be called a *neighbourhood of the point* c. In particular, the interval $(c - \varepsilon, c + \varepsilon)$ ($\varepsilon > 0$) is called the *ε-neighbourhood of the point* c.

Let $X = \{x\}$ be an arbitrary set of real numbers. The set X is said to be *bounded above* if $\exists$ a (real) number M such

3—01779

that $\forall x \in X : x \leqslant M$; to be *bounded below* if $\exists$ a number m such that $\forall x \in X$: $x \geqslant m$; and to be *bounded* if it is bounded both above and below.

We may, obviously, say that the set X is bounded if $\exists$ a number $M > 0$ such that $\forall x \in X$: $|x| \leqslant M$, since the inequality $|x| \leqslant M$ is equivalent to the following two inequalities: $-M \leqslant x \leqslant M$.

If a set X is not bounded, then it is called *unbounded*. It can be defined in the following way: the set X of real numbers is unbounded if $\forall M > 0$, $\exists x_0 \in X$: $|x_0| > M$. We can arrive at this statement, proceeding from the rule for constructing the negation of a given logical formula.

Examples. The closed interval $[a, b]$ is a bounded set. The open interval (a, b) is a bounded set if a and b are finite, and an unbounded set if $a = -\infty$ or $b = \infty$.

Sec. 1.11. Countable Sets. Countability of the Set of Rational Numbers. Uncountability of the Set of Real Numbers

We defined above the notion of equality of sets. In order to characterize the degree of saturation of infinite sets with elements, it is suitable to make use of the notion of equivalence of sets. A set X is said to be *infinite* if $\forall n \in \mathbf{N}$: the set X contains elements whose number exceeds n. Two sets A and B are called equivalent (written $A \sim B$) if it is possible to set up one-to-one correspondence ($\leftrightarrow$) between their elements, i.e. there exists such a rule (law) under which to $\forall a \in A$ there corresponds quite a definite element $b \in B$. By virtue of this rule, to two different elements $a_1, a_2 \in A$ there correspond two different elements $b_1, b_2 \in B$ and every element $b \in B$ corresponds to a certain element $a \in A$.

For instance, if A is the set of points of a circle of radius r, and B is the set of points on a concentric circle of radius $R > r$, then it is obvious that $A \sim B$ (Fig. 6). Obviously, if $A = B$, then $A \sim B$.

If $X = \{x\} \sim \mathbf{N} = \{n\}$, then the set X is called *countable*. It appears natural that the set of natural numbers $\mathbf{N}$ itself is countable (the correspondence being established according to the scheme $n \leftrightarrow n$). The set of all even natural

numbers $\mathsf{N}_e = \{2n\}$ is equivalent to the entire set N, the correspondence being set up according to the scheme $n \leftrightarrow 2n$. Note that here $\mathsf{N}_e \neq \mathsf{N}$, $\mathsf{N}_e \subset \mathsf{N}$. Thus, the true subset (portion) of a set turns out to be equivalent to the entire set. This property is inherent only in infinite sets (it may be taken for the definition of an infinite set).

It follows from the countability of a set that its elements can be numbered with the aid of natural numbers, therefore we shall frequently write a countable set in the form of a sequence of its elements:

$$X = \{x_1, x_2, \ldots, x_n, \ldots\}.$$

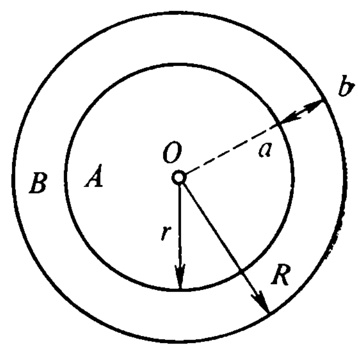

Fig. 6

A countable (set-theoretic) sum (union) of sets of the type

$$E = \bigcup_{k=1}^{\infty} E^k = E^1 + E^2 + \ldots$$

where each set E^k is countable is also a countable set. Indeed, let us arrange the elements $x_j^k \in E^k$ $(j = 1, 2, \ldots)$ in the form of a table:

$$E^1 = \{x_1^1, x_2^1, x_3^1, \ldots\},$$
$$E^2 = \{x_1^2, x_2^2, x_3^2, \ldots\},$$
$$E^3 = \{x_1^3, x_2^3, x_3^3, \ldots\},$$
$$\ldots\ldots\ldots\ldots\ldots$$

Now we can number them in the following order:

$$x_1^1, x_2^1, x_1^2, x_3^1, x_2^2, x_1^3, x_4^1, \ldots .$$

Here at each stage of the numbering process we should delete those elements that have already been numbered at the foregoing stages; the matter is that E^k and E^l may have some common elements. This procedure results in an infinite sequence of elements $\{y_1, y_2, y_3, \ldots\}$ which, obviously, exhausts the set E. This proves that E is a countable set.

We can similarly prove that a *finite sum* $E = E^1 + \dots + E^n$ *of countable or finite sets, among which there is at least one countable set is also countable.*

Theorem 1. *The set of all rational numbers is countable.*

Proof. Let us first consider the positive rational numbers $Q_+ = \{p/q\}$. The natural number $p + q$ will be called the height of the rational number p/q. Let now A_n be the set of all rational numbers with height equal to n. The sets A_n consist of a finite number of elements (rational numbers), for instance,

$$A_1 = \varnothing, \; A_2 = \left\{\frac{1}{1}\right\}, \; A_3 = \left\{\frac{1}{2}, \frac{2}{1}\right\},$$
$$A_4 = \left\{\frac{1}{3}, \frac{2}{2}, \frac{3}{1}\right\}, \; \dots .$$

It is easy to see that $Q_+ = \bigcup\limits_{n=1}^{\infty} A_n$.

Let us number the elements written between the braces from left to right, deleting, however, at each stage of the numbering process those elements which were numbered at the previous stages. As a result, we get the sequence

$$r_1 = 1, \quad r_2 = \frac{1}{2}, \quad r_3 = 2, \quad r_4 = \frac{1}{3}, \quad r_5 = 3, \; \dots .$$

Since there are indefinitely many rational positive numbers, we shall use all natural numbers. Hence, Q_+ is countable. Further, it is obvious that $Q_- = \{-p/q\}$ is countable. Therefore, the entire set of rational numbers $Q = Q_+ \cup Q_- \cup \{0\}$ is also countable.

Theorem 2. *The set of all real numbers is uncountable.*

Proof. To show this, it is sufficient to prove that the real numbers of the interval (0, 1) form an uncountable set. Suppose the contrary, i.e. that the interval (0, 1) is a countable set, that is, all its points can be numbered:

$$x_1 = 0 \cdot a_1^{(1)} a_2^{(1)} \dots .$$
$$\dots\dots\dots\dots$$
$$x_n = 0 \cdot a_1^{(n)} a_2^{(n)} \dots$$
$$\dots\dots\dots\dots$$

But this supposition is contradictory. Indeed, let us construct the real number $x = 0.a_1a_2 \dots$, where the digits a_n are chosen so that $0 < a_n < 9$ and $a_n \neq a_n^{(n)}$. It is clear that $x \in (0, 1)$, but x coincides with none of the numbers x_{n_0}, since, otherwise, there must be $a_{n_0} = a_{n_0}^{(n_0)}$ which is impossible.

CHAPTER 2

LIMIT OF SEQUENCE

Sec. 2.1. Concept of Limit of Sequence

Suppose that there is a certain law of correspondence which assigns to every natural number $n = 1, 2, 3, \ldots$ a real or a complex number x_n. Then we say that this defines a *sequence of numbers* $x_1, x_2, x_3, \ldots$ or, briefly, a sequence

$$\{x_n\} = \{x_1, x_2, x_3, \ldots\}.$$

We also say that the variable x_n runs through the sequence of values $\{x_n\}$.

The numbers x_n entering into a sequence $\{x_n\}$ are called its *terms*. It should be borne in mind that x_n and x_m for $n \neq m$ are regarded as different terms of a sequence, although it is not excluded that, as numbers, they are equal to each other, i.e. it may occur that $x_n = x_m$.

This chapter will deal with sequences of *real* numbers only.

Given below are examples of sequences:

Example 1. $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} = \left\{\frac{1}{n}\right\}$.

Example 2. $\left\{\frac{1}{2}, 2, \frac{1}{2}, 2, \ldots\right\} = \{2^{(-1)^n}\}$.

Example 3. $\left\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, \ldots\right\} = \{n^{(-1)^n}\}$.

Example 4. $\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\} = \left\{\frac{n-1}{n}\right\}$.

Example 5. $\{2, 5, 10, \ldots\} = \{n^2 + 1\}$.

Example 6. $\{-1, 2, -3, 4, \ldots\} = \{(-1)^n n\}$.

In Example 2 the variable x_n takes on one and the same value for all even n's:

$$2 = x_2 = x_4 = x_6 = \ldots.$$

Nevertheless we hold that the elements $x_2, x_4, \ldots$ are different.

A sequence $\{x_n\}$ is called *constant* if all its elements are equal to one and the same number a.

It is easily seen that the sequences in Examples 1, 2, and 4 are *bounded* (see Sec. 1.6). In such cases we also say that the corresponding variables running through these sequences are bounded. As to the sequences of Examples 3, 5, and 6, they are unbounded. But the sequence in Example 3 is, obviously, bounded by the number 0, and the sequence in Example 5 is bounded below by the number 2. As far as the last sequence is concerned, it is unbounded both below and above.

Let us now introduce the concept of the limit of a sequence.

Definition 1. *A number a is called the limit of a sequence $\{x_n\}$ if, given any arbitrary positive number ε, there can be found a natural number $n_0 = n_0(\varepsilon)$ (dependent on ε) such that the inequality*

$$|x_n - a| < \varepsilon \tag{1}$$

holds for all (natural) $n > n_0$.

In this case we write

$$\lim_{n\to\infty} x_n = \lim x_n = a \quad \text{or} \quad x_n \to a$$

and say that the variable x_n or the sequence $\{x_n\}$ has a limit equal to the number a, or tends to a. We also say that the variable x_n or the sequence $\{x_n\}$ *converges to the number a.*

If $x_n = a$, $\forall n \in \mathbf{N}$, then, obviously, $\lim\limits_{n\to\infty} x_n = \lim a = a$.

Remark. If $\lim x_n = a$, then $\lim x_{n+1} = a$; and conversely. This follows from the fact that if

$$|x_n - a| < \varepsilon, \quad \forall n > n_0,$$

then

$$|x_{n+1} - a| < \varepsilon, \quad \forall n > n_0 - 1,$$

and conversely.

The variable of Example 1 has the limit equal to 0:

$$\lim_{n\to\infty} \frac{1}{n} = 0. \tag{2}$$

Indeed, let there be given an arbitrary $\varepsilon > 0$ and let us solve the inequality

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon \quad \text{or} \quad \frac{1}{\varepsilon} < n.$$

Thus, for any $\varepsilon > 0$ we have found a number $n_0 = n_0(\varepsilon) = 1/\varepsilon$ such that the inequality

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$$

is fulfilled for all $n > n_0$, and we have proved equality (2).

Example 7. The variable of Example 4 tends to 1:

$$\lim_{n\to\infty} \frac{n-1}{n} = 1. \tag{3}$$

Indeed, let us set up the inequality

$$\left|1 - \frac{n-1}{n}\right| = \frac{1}{n} < \varepsilon.$$

As we saw, it is fulfilled for any $\varepsilon > 0$, provided that $n > n_0 = 1/\varepsilon$. This proves equality (3).

Example 8. If $|q| < 1$, then

$$\lim_{n\to\infty} q^n = 0. \tag{4}$$

Indeed, let for the time being $q \neq 0$. The inequality

$$|q^n - 0| = |q^n| < \varepsilon$$

is true if

$$n \log |q| < \log \varepsilon,$$

i.e. if

$$n > \frac{\log \varepsilon}{\log |q|} = n_0(\varepsilon).$$

We have proved (4) for $0 < |q| < 1$. If $q = 0$, then equality (4) is trivial, since in this case the variable q^n is a constant equal to zero:

$$\{0, 0, 0, \ldots\}.$$

Example 9. Expand a positive number a into a nonterminating decimal:

$$a = a_0.a_1a_2a_3 \ \ldots .$$

For its nth truncation

$$a^{(n)} = a_0.a_1 \ldots a_n$$

the following equality takes place:

$$\lim_{n\to\infty} a^{(n)} = a. \tag{5}$$

Indeed,

$$|a-a^{(n)}| = \underbrace{0.0 \ldots 0}_{n \text{ times}}a_{n+1}a_{n+2} \ldots \leqslant 10^{-n}.$$

But, given $\varepsilon > 0$, we can always find an n_0 such that

$$10^{-n} < \varepsilon, \quad \forall n > n_0$$

(see the preceding example, where we have to set $q = 10^{-1}$). Therefore

$$|a - a^{(n)}| < \varepsilon, \quad \forall n > n_0,$$

and, hence, we have proved (5).

Remark. The truncations $a^{(n)}$ $(n = 1, 2, \ldots)$ are rational numbers. It follows from (5) that *any real number* is the limit of sequence of rational numbers.

Thus, any irrational number can be approximated by a rational number with any preassigned degree of accuracy.

By virtue of this property, the set $\mathbf{Q}$ of rational numbers is said to be *dense everywhere* in the set $\mathbf{R}$ of all real numbers.

The inequality

$$|x_n - a| < \varepsilon$$

is equivalent to the two inequalities

$$-\varepsilon < x_n - a < \varepsilon \quad \text{or} \quad a - \varepsilon < x_n < a + \varepsilon,$$

which is equivalent to the fact that the point x_n belongs to the ε-neighbourhood of the point a:

$$x_n \in (a - \varepsilon, a + \varepsilon)$$

(see Sec. 1.10).

Therefore we can restate the definition of a limit in geometrical terms: a variable x_n has as its limit a number (point) a if, given any $\varepsilon > 0$, there is a natural number n_0 such that all the points x_n with $n > n_0$ will be contained in the ε-neighbourhood of the point a:

$$x_n \in (a - \varepsilon, a + \varepsilon) \quad (n > n_0).$$

It is obvious that for an arbitrary neighbourhood (c, d) of the point a we can find $\varepsilon > 0$ such that the interval $(a - \varepsilon, a + \varepsilon)$ is contained in (c, d), that is $(a - \varepsilon, a + \varepsilon) \subset (c, d)$ (Fig. 7).

Therefore the fact that $x_n \to a$ can also be expressed as follows: given an arbitrary neighbourhood (c, d) of the

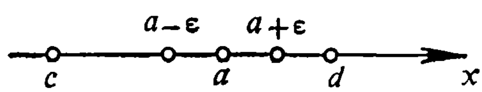

Fig. 7

point a, all the points x_n, beginning with some n, are in that neighbourhood, i.e. there must exist a number n_0 such that $x_n \in (c, d)$ $(n > n_0)$. As to the points x_n with $n \leqslant n_0$, they may or may not belong to (c, d). Hence, if there are some points x_n lying outside (c, d), then their number is finite.

On the other hand, if it is known that there are only a finite number of points $x_{n_1}, x_{n_2}, \ldots, x_{n_s}$ lying outside (c, d), we can put

$$k = \max \{n_1, n_2, \ldots, n_s\},$$

and then $x_n \in (c, d)$ for all $n > k$. Thus, the definition of limit can also be stated as follows: a variable x_n has as its limit a number a if the *set of points x_n lying outside any neighbourhood of the point a is either finite or empty.*

Example 10. The variable

$$\{(-1)^{n+1}\} = \{1, -1, 1, -1, \ldots\} \tag{6}$$

has no limit at all.

Indeed, let us assume that this variable has a limit equal to a. Consider the neighbourhood of this point

$$\left(a-\frac{1}{3},\ a+\frac{1}{3}\right).$$

Its length is equal to 2/3. Obviously, this neighbourhood cannot contain simultaneously both the point 1 and the point -1, since the distance between these points equals 2 $(2 > 2/3)$. For the sake of definiteness, we shall hold that the point 1 does not belong to our neighbourhood.

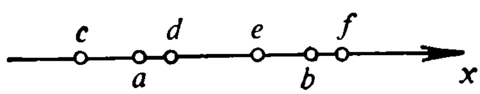

Fig. 8

But $x_n = 1$ for $n = 1, 3, 5, \ldots$, i.e. there are infinitely many elements of the sequence outside our neighbourhood.

Hence, the point a cannot be the limit of our sequence, and since this point is an arbitrary one, sequence (6) has no limit.

Theorem 1. *If a variable x_n has a limit, then it is unique.*

Indeed, let us assume, in spite of this statement, that x_n has two different limits a and b. Let us "surround" the points a and b respectively with the intervals (c, d) and (e, f) each of a length so small as not to intersect each other (see Fig. 8). Since $x_n \to a$, the interval (c, d) would contain all the elements x_n except possibly a finite number of them, but then the interval (e, f) cannot contain an infinite number of elements x_n and x_n cannot tend to b. We have arrived at a contradiction. The theorem has been proved.

Theorem 2. *If a sequence $\{x_n\}$ converges, then it is bounded.*

Proof. Let $\lim x_n = a$. We then set $\varepsilon = 1$ and choose a natural $n_0 = n_0\ (1)$ so that

$$1 > | x_n - a | \quad (n > n_0),$$

but then $1 > | x_n | - | a |$ and the inequality

$$1 + | a | > | x_n |$$

is fulfilled for all $n > n_0$. Let M be the greatest number among

$$1 + |a|, \ |x_1|, \ |x_2|, \ \ldots, \ |x_{n_0}|$$

Then, obviously,

$$M \geqslant |x_n|, \quad \forall n \in \mathbf{N}.$$

The theorem has been proved.

Remark. As is shown in Example 10, the boundedness of a sequence is a necessary, but not sufficient, condition for a sequence to converge.

Theorem 3. *If a variable x_n has a nonzero limit a, then we can find an n_0* such that

$$|x_n| > |a|/2 \quad \text{for} \quad n > n_0.$$

Moreover, for these n's, if $a > 0$ then $x_n > a/2$ and if $a < 0$, then $x_n < a/2$. Thus, beginning with some value of n, the variable x_n is of the same sign as a.

Proof. Let $x_n \to a$. Then for $\varepsilon = |a|/2$ there exists a number n_0 such that

$$|a|/2 > |a - x_n| \geqslant |a| - |x_n| \quad (n > n_0),$$

whence $|x_n| > |a| - \frac{|a|}{2} = \frac{|a|}{2}$, and the first assertion of the theorem has thus been proved. On the other hand, the inequality $|a|/2 > |a - x_n|$ is equivalent to the two inequalities

$$a - \frac{|a|}{2} < x_n < a + \frac{|a|}{2} \quad (n > n_0).$$

Then if $a > 0$, then

$$x_n > \frac{a}{2} = a - \frac{|a|}{2} \quad (n > n_0),$$

and if $a < 0$, then

$$x_n < a + \frac{|a|}{2} = a - \frac{a}{2} = \frac{a}{2} \quad (n > n_0),$$

which completes the proof on the second assertion of the theorem.

Theorem 4. *If $x_n \to a$, $y_n \to b$* and $x_n \leqslant y_n$ *for all* $n = 1, 2, \ldots$, then $a \leqslant b$.

Proof. Suppose that $b < a$. Let us set a number $\varepsilon < (a - b)/2$ $(\varepsilon > 0)$ and find natural numbers N_1 and N_2 such that

$$a - \varepsilon < x_n \quad (n > N_1), \quad y_n < b + \varepsilon \quad (n > N_2).$$

This is possible since $x_n \to a$ and $y_n \to b$.

If $n_0 = \max\{N_1, N_2\}$, then, obviously, $y_n < b + \varepsilon < a - \varepsilon < x_n$ $(n > n_0)$, and thus we arrive at a contradiction because, by the hypothesis, $x_n \leqslant y_n$ for all n's.

Corollary. *If the elements of a convergent sequence $\{x_n\}$ belong to a closed interval $[a, b]$, then its limit also belongs to ' this interval.*

Proof. Indeed, $a \leqslant x_n \leqslant b$. If $\lim x_n = c$, by Theorem 4, $a \leqslant c \leqslant b$, which is what we set out to prove.

Remark. For the interval (a, b) we may only assert that if $x_n \in (a, b)$, then $\lim x_n = c \in [a, b]$.

Thus, inequalities, in the limit, are retained or turn into an equality. For instance, $x_n = 1/(n + 1) \in (0, 1)$, but $c = 0 \in [0, 1]$.

Theorem 5. *If variables x_n and y_n tend to one and the same limit a and if $x_n \leqslant z_n \leqslant y_n$ $(n = 1, 2, \ldots)$, then the variable z_n also tends to a.*

Proof. Given $\varepsilon > 0$, we can find N_1 and N_2 such that

$$a - \varepsilon < x_n \quad (n > N_1), \quad y_n < a + \varepsilon \quad (n > N_2),$$

whence for $n > n_0 = \max\{N_1, N_2\}$

$$a - \varepsilon < x_n \leqslant z_n \leqslant y_n < a + \varepsilon$$

and

$$|z_n - a| < \varepsilon \quad (n > n_0),$$

which completes our proof.

Theorem 6. *If $x_n \to a$, then $|x_n| \to |a|$.*

The proof follows from the inequality $||x_n| - |a|| \leqslant |x_n - a|$.

Sec. 2.2. Arithmetic Operations with Variables Having a Limit

Let x_n and y_n be variables which respectively run through the sequences $\{x_n\}$ and $\{y_n\}$. The *sum* $x_n + y_n$, the *difference* $x_n - y_n$, the *product* $x_n y_n$, and the *quotient*

x_n/y_n of x_n and y_n are defined as *variables* running respectively through the sequences $\{x_n + y_n\}$, $\{x_n - y_n\}$, $\{x_n y_n\}$, and $\{x_n/y_n\}$. In the case of quotient it is assumed that $y_n \neq 0$ for all $n = 1, 2, \ldots$.

If $x_n = c$ for $n = 1, 2, \ldots$, then we write $c \pm y_n$, cy_n, c/y_n instead of $x_n \pm y_n$, $x_n y_n$, x_n/y_n.

There hold the following assertions:

$$\lim (x_n \pm y_n) = \lim x_n \pm \lim y_n, \tag{1}$$

$$\lim (x_n y_n) = \lim x_n \cdot \lim y_n, \tag{2}$$

$$\lim \frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n}, \quad \text{if} \quad \lim y_n \neq 0. \tag{3}$$

These assertions should be understood in the sense that *if x_n and y_n possess finite limits, then the limits of their sum, difference, product, and quotient also exist (in the latter case under the assumption that* $\lim y_n \neq 0$) *and equalities* (1) *to* (3) *hold.*

Proof. Let $x_n \to a$ and $y_n \to b$. Setting an arbitrary $\varepsilon > 0$, we choose n_0 so that

$$|x_n - a| < \varepsilon/2, \quad |y_n - b| < \varepsilon/2 \quad (n > n_0).$$

Then

$$|(x_n \pm y_n) - (a \pm b)| \leqslant |x_n - a| + |y_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (n > n_0),$$

and we have thus proved (1).

To prove (2), note that

$$|x_n y_n - ab| = |x_n y_n - a y_n + a y_n - ab|$$
$$\leqslant |x_n y_n - a y_n| + |a y_n - ab| = |y_n|\,|x_n - a| + |a|\,|y_n - b|. \tag{4}$$

Since y_n has a limit (by Theorem 2 proved in the preceding section) there exists a positive number M such that

$$|y_n| \leqslant M \quad (n = 1, 2, \ldots), \tag{5}$$

$$|a| \leqslant M. \tag{6}$$

We choose n_0 so that

$$|x_n - a| < \frac{\varepsilon}{2M}, \quad |y_n - b| < \frac{\varepsilon}{2M} \quad (n > n_0). \tag{7}$$

Then from (4)-(7) it follows that

$$|x_n y_n - ab| < \frac{M\varepsilon}{2M} + \frac{M\varepsilon}{2M} = \varepsilon \quad (n > n_0).$$

This proves equality (2).

Let us add (to the conditions $x_n \to a$ and $y_n \to b$) the supplementary condition $b \neq 0$. Then

$$\left|\frac{x_n}{y_n} - \frac{a}{b}\right| = \left|\frac{x_n b - y_n a}{y_n b}\right| = \frac{|(x_n - a)\,b + (b - y_n)\,a|}{|y_n|\;|b|}$$
$$\leqslant \frac{|x_n - a|}{|y_n|} + \frac{|b - y_n|\;|a|}{|y_n|\;|b|}. \tag{8}$$

Now it is convenient to use Theorem 3 of the preceding section, according to which

$$|\,y_n\,| > |\,b\,|/2 \quad (n > N_1) \tag{9}$$

for sufficiently large N_1. Setting an arbitrary $\varepsilon > 0$, we choose N_2 and N_3 such that

$$|\,x_n - a\,| < \varepsilon\,|\,b\,|/4 \quad (n > N_2), \tag{10}$$

and

$$|\,a\,|\,|\,y_n - b\,| < \varepsilon b^2/4 \quad (n > N_3). \tag{11}$$

Then, setting $n_0 = \max\,\{N_1, N_2, N_3\}$, by virtue of (8)-(11), we shall have

$$\left|\frac{x_n}{y_n} - \frac{a}{b}\right| < \frac{\varepsilon \cdot |b|}{4} \cdot \frac{2}{|b|} + \frac{\varepsilon}{2} = \varepsilon \quad (n > n_0),$$

which proves equality (3).

Note that the limits of the variables on the left-hand sides of equalities (1) to (3) may exist even when the limit of x_n or the limit of y_n (or both) does not exist. For instance, if $x_n = (-1)^n$, $y_n = (-1)^{n+1}$, then x_n and y_n have no limits, while $\lim\,(x_n + y_n) = 0$, and $\lim x_n y_n = -1$.

The theorems on the limits of sums, differences, products, and quotients make it possible to find out in many cases whether a given variable has a limit and what this limit is, provided that this variable is the result of a finite number of arithmetic operations on some other variables, the existence of whose limits is established and the values of the limits are known.

But there are also many cases when the above theorems are inapplicable. To find such limits a mathematician should apply some other methods.

Example 1. Let $x_n = 1 + q + \ldots + q^n$, $|q| < 1$. Prove that

$$\lim_{n\to\infty} x_n = \frac{1}{1-q}.$$

We have

$$x_n = \frac{1-q^{n+1}}{1-q} = \frac{1}{1-q} - \frac{q^{n+1}}{1-q}.$$

Since $\lim q^{n+1} = 0$ for $|q| < 1$, using formulas (1) and (2), we get

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{1-q} - \frac{1}{1-q} \lim_{n\to\infty} q^{n+1}$$

$$= \frac{1}{1-q} - \frac{1}{1-q}\cdot 0 = \frac{1}{1-q}.$$

Hence forward, by the notation

$$1+q+\ldots+q^n+\ldots \equiv \sum_{n=0}^{\infty} q^n$$

we shall understand $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} q^k$. Thus,

$$\sum_{n=0}^{\infty} q^n = \lim_{n\to\infty} x_n = \frac{1}{1-q} \quad (|q| < 1).$$

Sec. 2.3. Infinitesimals and Infinities

A variable α_n having a zero limit is called an *infinitely small quantity* (magnitude) or, simply, an *infinitesimal*.

Thus, a variable α_n is an infinitesimal if for any $\varepsilon > 0$ we can find n_0 such that $|\alpha_n| < \varepsilon$ $(n > n_0)$.

It is easy to see that *for a variable x_n to have a limit a it is necessary and sufficient that $x_n = a + \alpha_n$, where α_n is an infinitesimal.*

A variable β_n is called an *infinitely large quantity* (magnitude) or, simply, an *infinity*, if for any $M > 0$

there can be found n_0 such that $|\beta_n| > M$ $(n > n_0)$. For an infinitely large variable β_n we write

$$\lim \beta_n = \infty, \quad \text{or} \quad \beta_n \to \infty \tag{1}$$

and say that β_n *tends to infinity*.

If, beginning with some n_0, an infinity β_n assumes only positive values or only negative values, then we write, respectively,

$$\lim \beta_n = +\infty, \quad \text{or} \quad \beta_n \to +\infty, \tag{2}$$

$$\lim \beta_n = -\infty, \quad \text{or} \quad \beta_n \to -\infty. \tag{3}$$

Thus, from (2) as also from (3), follows (1). The example of the variable $\{(-1)^n n\}$ shows that relation (1) may hold, while neither (2), nor (3) is valid.

We state the following obvious properties:

1. *If a variable x_n is bounded and y_n is infinitely large, then $x_n/y_n \to 0$.*

2. *If the absolute value of x_n is bounded below by a positive number and y_n is a nonzero infinitesimal, then $x_n/y_n \to \infty$.*

We shall confine ourselves to the proof of the second property.

By the hypothesis, there is a number $a > 0$ such that the inequality $|x_n| > a$ holds for $n = 1, 2, \ldots$, and for any $\varepsilon > 0$ there exists n_0 such that

$$|y_n| < \varepsilon \quad (n > n_0). \tag{4}$$

Then

$$\left|\frac{x_n}{y_n}\right| > \frac{a}{\varepsilon} = M \quad (n > n_0).$$

Taking an arbitrary positive number M, we can choose ε such that $M = a/\varepsilon$ and then find, for the given ε, a natural n_0 so that Property (4) holds. Then $|x_n/y_n| > M$ $(n > n_0)$, which is what we intended to prove.

The above two assertions imply the following corollaries:

$$\lim_{y_n \to \infty} \frac{c}{y_n} = 0, \quad \lim_{y_n \to 0} \frac{c}{y_n} = \infty \quad (c \neq 0).$$

Note that if a sequence $\{x_n\}$ is unbounded, then it is not necessarily infinitely large. For instance, the sequence

$$\{n^{(-1)^n}\}=\left\{1,\ 2,\ \frac{1}{3},\ 4,\ \ldots\right\}$$

is unbounded, but it is not infinitely large, since it contains arbitrarily small terms with an arbitrarily large (odd) number.

Remark. Any nonzero constant quantity (sequence) is not an infinitesimal. Of all constant quantities, only one, which is equal to zero, is infinitely small. If a certain quantity is known to be constant and its absolute value is less than any positive number ε, then it is equal to zero.

Theorem 1. *The product of an infinitely small sequence by a bounded one is an infinitely small sequence, i. e. if*

$\lim x_n = 0$ *and* $|y_n| \leqslant M$, $\forall n \in \mathbf{N}$, *then* $\lim x_n y_n = 0$.

Indeed, let us take $\varepsilon > 0$ and choose n_0 so that

$$|x_n| < \varepsilon/M, \quad \forall n > n_0.$$

Then

$$|x_n y_n - 0| = |x_n|\,|y_n| \leqslant \frac{\varepsilon}{M} M = \varepsilon, \quad \forall n > n_0,$$

which completes the proof.

Sec. 2.4. Evaluation of Indeterminate Forms

1. Let $\lim x_n = \lim y_n = 0$ $(y_n \neq 0)$.

Consider the sequence $\{x_n/y_n\}$. Nothing definite can be said about the limit of this sequence in advance. This is shown by the following particular examples:

if $x_n=\frac{1}{n}$, $y_n=\frac{1}{n^2}$, then $\frac{x_n}{y_n}=n\to+\infty$ for $n\to\infty$;

if $x_n=\frac{1}{n^2}$, $y_n=\frac{1}{n}$, then $\frac{x_n}{y_n}=\frac{1}{n}\to 0$ for $n\to\infty$;

if $x_n=\frac{a}{n}$, $y_n=\frac{1}{n}$, then $\frac{x_n}{y_n}=a\to a$ for $n\to\infty$;

if $x_n=\frac{(-1)^n}{n}$, $y_n=\frac{1}{n}$, then $\frac{x_n}{y_n}=(-1)^n$ and the limit of this sequence does not exist at all.

Thus to find the limit of $\{x_n/y_n\}$, it is not sufficient to know that $x_n \to 0$, $y_n \to 0$. Some additional information is necessary to characterize the variation of x_n and y_n. Special technique is required to find this limit in each particular case.

The ratio x_n/y_n, for $x_n \to 0$, $y_n \to 0$ is said to represent an *indeterminate form of the type* $\frac{0}{0}$.

2. If $x_n \to \infty$, $y_n \to \infty$, then the expression x_n/y_n also represents an indeterminate form called the *indeterminate form of the type* $\frac{\infty}{\infty}$.

3. If $x_n \to 0$, $y_n \to \infty$, then for the expression $x_n y_n$ we obtain an *indeterminate form of the type* $0 \cdot \infty$.

4. If $x_n \to +\infty$, $y_n \to -\infty$, then the expression $x_n + y_n$ represents an *indeterminate form of the type* $\infty - \infty$.

Each of the above cases can be illustrated by particular examples.

To evaluate a certain indeterminate form means to find the limit (provided that it exists) of the corresponding expression which is, however, not always an easy thing.

Example 1. If

$$x_n = a_m n^m + \ldots + a_1 n + a_0,$$

$$y_n = b_l n^l + \ldots + b_1 n + b_0 \quad (a_m \neq 0,\ b_l \neq 0),$$

then for $n \to \infty$ the expression x_n/y_n represents an indeterminate form of the type $\frac{\infty}{\infty}$. Let us evaluate this indeterminate form.

(a) If $l = m$, then, dividing both the numerator and denominator by n^m, we get

$$\frac{x_n}{y_n} = \frac{a_m + \frac{a_{m-1}}{n} + \ldots + \frac{a_0}{n^m}}{b_m + \frac{b_{m-1}}{n} + \ldots + \frac{b_0}{n^m}} \to \frac{a_m}{b_m}$$

for $n \to \infty$, i.e. $\lim (x_n/y_n) = a_m/b_m$, i.e. equal to the ratio of the coefficients of the highest powers of n in the expressions for x_n and y_n.

(b) Analogously, we can show that for $m > l$ $\lim (x_n/y_n) = \infty$, and for $m < l$ $\lim (x_n/y_n) = 0$.

Example 2. If $x_n = \sqrt{n+1}$, $y_n = \sqrt{n}$, then for $n \to \infty$ the expression $x_n - y_n$ yields an indeterminate form of the type $\infty - \infty$.

Let us evaluate this indeterminate form:

$$x_n - y_n = \sqrt{n+1} - \sqrt{n} = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} (\sqrt{n+1} + \sqrt{n})$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0$$

for $n \to \infty$. Hence $\lim\limits_{n \to \infty} (\sqrt{n+1} - \sqrt{n}) = 0$.

Sec. 2.5. Monotone Sequences

Definition. A sequence $\{x_n\}$ is called *nondecreasing* (*nonincreasing*) if $\forall n \in \mathbb{N}$ the inequality

$$x_n \leqslant x_{n+1} \quad (x_n \geqslant x_{n+1})$$

holds true.

If the absolute inequalities $x_n < x_{n+1}$ $(x_n > x_{n+1})$ are really fulfilled, then the sequence $\{x_n\}$ is called *absolutely increasing* (*absolutely decreasing*) or, simply, *increasing* (*decreasing*). Decreasing and increasing, nondecreasing and nonincreasing sequences are called *monotonic* (or monotone).

The terms of monotonic sequences can be arranged in small chains $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant x_{n+1} \leqslant \ldots$ $(x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant x_{n+1} \geqslant \ldots)$, whence it is seen that a nondecreasing sequence is bounded below, and a nonincreasing one is bounded above.

Examples.

(1) $\left\{1, 1, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{n}, \frac{1}{n}, \ldots\right\}$ is a nonincreasing sequence;

(2) $\{n^2\}$ is an increasing sequence.

Below, we prove an important theorem which states that a monotonic bounded sequence of numbers always

has a limit. In Sec. 1.7 this theorem appeared as one of the basic properties (Property V) of the set of real numbers.

Theorem 1. *If a sequence of real numbers*

$$a_1, \ a_2, \ a_3, \ldots \tag{1}$$

does not decrease (does not increase) and is bounded above (below) by a number M (respectively, by m), then there exists a real number a such that it is not greater than M (not less than m) to which this sequence tends as to its limit:

$$\lim_{n\to\infty} a_n = a \leqslant M \tag{2}$$

($\lim\limits_{n\to\infty} a_n = a \geqslant m$, *respectively*).

Proof. Let sequence (1) does not decrease and let for the time being $a_1 > 0$, then all $a_n > 0$ $(n = 1, 2, 3, \ldots)$. We then expand each term of the sequence into a nonterminating decimal:

$$a_n = a_{n0}.\, a_{n1} a_{n2} a_{n3} \ldots \quad (n = 1, 2, \ldots). \tag{3}$$

Since the sequence $\{a_n\}$ is bounded above by the number M $(a_n \leqslant M)$ and does not decrease, then, according to Lemma 2 considered in Sec. 1.6, decimals (3) are stabilized to a certain number $a \leqslant M$:

$$a_n \rightrightarrows a = \gamma_0 . \gamma_1 \gamma_2 \ldots,$$

but then a_n tends to a as to its limit:

$$\lim_{n\to\infty} a_n = a.$$

Indeed, for any ε there can be found a natural m such that $10^{-m} < \varepsilon$. Since a_n is stabilized to a, we have

$$a_n = \gamma_0 . \gamma_1 \ldots \gamma_m a_{n,\, m+1}\, a_{n,\, m+2} \ldots$$

for all $n > n_0$, where n_0 is sufficiently large, but then

$$|a - a_n| = a - a_n \leqslant \underbrace{0.0 \ldots 0}_{m \text{ times}} \gamma_{m+1} \gamma_{m+2} \ldots \leqslant 10^{-m} < \varepsilon$$

$$(n > n_0),$$

i.e. $a_n \to a$ for $n \to \infty$.

If $a_1 \leqslant 0$, then we add to a_1 a number c large enough to get $a_1 + c > 0$ and set $b_n = a_n + c$ $(n = 1, 2, \ldots)$.

The sequence $\{b_n\}$ does not decrease, is bounded above by the number $M + c$ and its terms are positive. Therefore, according to the above proved theorem, there exists the limit $\lim\limits_{n\to\infty} b_n = b \leqslant \leqslant M + c$, but then there also exists the limit $\lim\limits_{n\to\infty} a_n = = \lim\limits_{n\to\infty} (b_n - c) = b - c \leqslant M$, and the theorem has been proved for an arbitrary nondecreasing sequence.

If now the sequence $\{a_n\}$ does not increase and is bounded below by a number m, then the sequence of numbers $\{-a_n\}$ does not decrease, is bounded above by the number $-m$, and, on the basis of what has been already proved, there exists the limit $\lim\limits_{n\to\infty} (-a_n) = = -a \leqslant -m$ which is denoted by $-a$. Consequently, there also exists $\lim\limits_{n\to\infty} a_n = -\lim\limits_{n\to\infty} (-a_n) = -(-a) = a \geqslant m$. The theorem has been proved.

Remark. If a sequence of real numbers $\{a_n\}$ is convergent, then their decimal expansions are not necessarily stabilized. For instance, if

$$a_{2k} = 1.\underbrace{0 \ldots 0}_{k \text{ times}}11 \ldots \quad (k = 1, 2, \ldots),$$

$$a_{2k+1} = 0.\underbrace{9 \ldots 9}_{k \text{ times}}11 \ldots \quad (k = 1, 2, \ldots),$$

and the sequence $\{a_n\}$ has a limit equal to 1 $(a_n \to 1)$, but, as it is easily seen, this sequence is not stabilized.

Example 1. Let us give a new proof of the equality

$$\lim_{n\to\infty} q^n = 0, \quad |q| < 1 \tag{4}$$

(cf. Example 8 given in Sec. 2.1).

Let for the time being $q \geqslant 0$. Then the variable q^n $(n = 1, 2, \ldots)$ does not increase and is bounded below by the number 0. But then, by Theorem 1, there exists a number $A \geqslant 0$ to which q^n tends:

$$\lim_{n\to\infty} q^n = A.$$

We also have

$$A = \lim_{n\to\infty} q^{n+1} = q \lim_{n\to\infty} q^n = qA,$$

whence $A(1 - q) = 0$ and $A = 0$, since $q < 1$.

If now $q < 0$, then, on the basis of what has been already proved, $|q^n| = |q|^n \to 0$, $n \to \infty$.

Equality (4) has been completely proved.

This proof of (4) is, perhaps, more elegant as compared with the proof given in Example 8 of Sec. 2.1, but it provides no possibility to judge about the rate at which q^n tends to zero, since we are not given effectively the number $n_0 = n_0(\varepsilon)$ beginning with which $|q^n| < \varepsilon$.

Example 2. There holds the equality

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0, \tag{5}$$

where a is an arbitrary number.

For $|a| \leqslant 1$ it is obvious. Let $a > 1$. We set

$$u_n = \frac{a^n}{n!}.$$

Then

$$\frac{u_{n+1}}{u_n} = \frac{a}{n+1} \to 0 \quad (n \to \infty).$$

Hence it follows that $u_{n+1} < u_n$, $\forall n > n_0$, where n_0 is sufficiently large.

Thus, the variable u_n decreases for $n > n_0$. Besides, it is bounded below by the number 0. But then there exists the limit

$$\lim_{n\to\infty} u_n = A \geqslant 0.$$

And also

$$A = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} \left(u_n \frac{a}{n+1}\right) = A \lim_{n\to\infty} \frac{a}{n+1} = A \cdot 0 = 0,$$

and we have proved equality (5) for any $a \geqslant 0$. But it is also true for any $a < 0$ since

$$\left|\frac{a^n}{n!}\right| = \frac{|a|^n}{n!} \to 0 \quad \text{as} \quad n \to \infty.$$

Sec. 2.6. The Number *e*

Consider the sequence

$$\{x_n\} = \left\{\left(1 + \frac{1}{n}\right)^n\right\}.$$

We are going to show that this sequence increases and is bounded above. By the binomial formula

$$(a+b)^n = \sum_{k=0}^{n} \frac{n(n-1)\ldots(n-k+1)}{k!} a^{n-k} b^k,$$

we have

$$x_n = \left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n}$$
$$+\dots+\frac{n(n-1)\dots(n-k+1)}{k!}\frac{1}{n^k}$$
$$+\dots+\frac{n(n-1)\dots(n-n+1)}{n!}\cdot\frac{1}{n^n}$$
$$=2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\dots+\frac{1}{k!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{k-1}{n}\right)$$
$$+\dots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{n-1}{n}\right). \qquad (1)$$

It is seen from this equality that the sequence $x_n \geqslant 2$, $\forall n$. Let us prove that the sequence $\{x_n\}$ is bounded above. From equality (1) we have

$$x_n \leqslant 2+\frac{1}{2!}+\dots+\frac{1}{n!}\leqslant 2+\frac{1}{2}$$
$$+\dots+\frac{1}{2^{k-1}}+\dots+\frac{1}{2^{n-1}}$$
$$\leqslant 1+\left(1+\frac{1}{2}+\dots+\frac{1}{2^n}+\dots\right)=1+\frac{1}{1-\frac{1}{2}}=3.$$

Let us show that the sequence $\{x_n\}$ is an increasing one. By analogy with (1), we have

$$x_{n+1}=2+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)$$
$$+\dots+\frac{1}{k!}\left(1-\frac{1}{n+1}\right)\dots\left(1-\frac{k-1}{n+1}\right)$$
$$+\dots+\frac{1}{n!}\left(1-\frac{1}{n+1}\right)\dots\left(1-\frac{n-1}{n+1}\right)$$
$$+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\dots\left(1-\frac{n}{n+1}\right). \qquad (2)$$

Comparing (1) with (2), we see that $x_n < x_{n+1}$, $\forall n \in \mathbf{N}$ (the terms in (2) are greater than the corresponding terms in (1) and, besides, (2) includes an extra positive term). By Theorem 1 proved in the preceding section, the sequence $\{x_n\}$ converges. Let us denote its limit by the

lower-case letter e (L. Euler* was the first to suggest this notation):

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

It is clear from the aforesaid that $2 < e < 3$. A more accurate value of e is given below:

$$e = 2.718281 \ldots .$$

Hereafter (in Sec. 4.16), we shall prove a formula from which it follows that

$$e = \sum_{k=0}^{n} \frac{1}{k!} + \frac{\theta}{n!} \quad (n > 2), \tag{3}$$

where θ is a certain number dependent on n and satisfying the inequalities $0 < \theta < 1$. Using this formula, it is easy to prove that e is an irrational number. Suppose, $e = p/q$, where p and q are natural numbers. Then, setting in (3) $n = q$, we shall have

$$\frac{p}{q} = \sum_{k=0}^{q} \frac{1}{k!} + \frac{\theta}{q!}.$$

Multiplying by $q!$, we get

$$p\,(q-1)! - l = \theta, \tag{4}$$

where $l = q! \sum_{k=0}^{q} \frac{1}{k!}$ is a natural number. We have arrived at a contradiction, since the left-hand member of (4) is a whole number, whereas the right-hand member equal to θ is a proper fraction.

Sec. 2.7. The Principle of Nested Intervals

Theorem 1 (the principle of nested intervals**). *Let there be given a sequence of nested intervals*

$$\sigma_n = [a_n, b_n] \quad (n = 1, 2, \ldots),$$

---

* Euler, Leonhard (1707-1783). Gifted Swiss mathematician, the most prolific mathematician in history, and the first modern mathematical universalist.

** A sequence of intervals such that each is contained in the preceding.

i.e. *such that* $\sigma_{n+1} \subset \sigma_n$ $(n = 1, 2, \ldots)$, *with lengths tending to zero*:

$$d_n = b_n - a_n \to 0 \quad (n \to \infty).$$

Then there exists a unique point c (number) *simultaneously belonging to all the intervals* σ_n $(c \in \sigma_n, n = 1, 2, \ldots)$.

Proof. It is obvious that

$$a_1 \leqslant a_2 \leqslant a_3 \leqslant \ldots \leqslant b_m$$

for any given natural m. This shows that the numbers a_n do not decrease and are bounded above by a number b_m for any m, and, according to Theorem 1 proved in Sec. 2.5, there exists a number c to which the variable a_n ($\lim a_n = c$) tends. Here $a_n \leqslant c \leqslant b_m$. Since natural n and m are arbitrary in these inequalities, then, in particular, $a_n \leqslant c \leqslant b_n$ $(n = 1, 2, \ldots)$. Consequently, $c \in \sigma_n$, whatever $n \in \mathbb{N}$ may be.

The found point c is unique. Let us assume that there exists another point $c_1 \in \sigma_n$, $\forall n$. Then $a_n \leqslant c$, $c_1 \leqslant \leqslant b_n$, whence

$$b_n - a_n \geqslant | c - c_1 | > 0, \quad \forall n,$$

but this contradicts the fact that $b_n - a_n \to 0$. Note that $\lim b_n = \lim [(b_n - a_n) + a_n] = c$.

Remark. It is essential that Theorem 1 deals with closed $[a_n, b_n]$ but not open intervals, as is illustrated by the following example. The open intervals $(0, 1/n)$ $(n = 1, 2, \ldots)$ form a nested family and their lengths tend to zero as n increases indefinitely $\left(d_n = \frac{1}{n} - 0 = \frac{1}{n} \to 0\right)$ but at the same time there is no point simultaneously belonging to all these intervals.

Indeed, any point $c \leqslant 0$ does not belong to any of the intervals $(0, 1/n)$. But if $c > 0$, then we can find an n such that $1/n < c$ and $c \bar{\in} (0, 1/n)$.

Sec. 2.8. Supremum and Infimum of a Set

Let us consider an arbitrary set E of real numbers x. It may happen that this set contains the greatest (maximum) number which will be denoted by M. In this case

we write

$$M = \max E = \max_{x \in E} x$$

It may also occur that among the numbers $x \in E$ there is the least (minimum) number equal to m. Then we write

$$m = \min E = \min_{x \in E} x.$$

If the set E is finite, i.e. it consists of a finite number of numbers

$$x_1, x_2, \ldots, x_p,$$

then among them there are always the greatest and the least ones.

But this is not always so if E is an infinite set. We shall clarify our assertion by considering the following examples:

(1) $\mathsf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$,

(2) $\mathsf{N} = \{1, 2, 3, \ldots\}$,

(3) $\mathsf{N}_- = \{\ldots, -2, -1\}$,

(4) $[a, b]$,

(5) $[a, b)$,

(6) (a, b).

The set Z has neither the greatest, nor the least number. The open interval (a, b) has no greatest and least numbers either. Here it is of no importance whether the numbers a, b are finite or infinite. Whatever the number $c \in (a, b)$ may be, i.e. the number satisfying the inequalities $a < c < b$, we can always find numbers c_1, c_2 such that $a < c_1 < c < c_2 < b$.

The set N has no greatest element, but does have the least one: $x = 1$. And the set N_- has the greatest element $x = -1$, but does not have the least one.

It is also obvious that $\min [a, b] = a$, $\max [a, b] = b$, $\min [a, b) = a$, but there is no maximum number in $[a, b)$.

There arises a question concerning the introduction for an arbitrary set E of numbers which, as far as possible, would substitute $\max E$ and $\min E$. Such numbers (fi-

nite or infinite) are: the *least upper bound* or *supremum*

$$\sup E = \sup_{x \in E} x = M$$

and the *greatest lower bound* or *infimum*

$$\inf E = \inf_{x \in E} x = m$$

of the set.

Let the set E be bounded above.

A number M (finite) is called the *least upper bound* or *supremum* of a set E if the following two conditions hold:

(1) $x \leqslant M, \ \forall x \in E$,

(2) for any $\varepsilon > 0$ there exists a point $x_1 \in E$ such that the following inequalities are fulfilled:

$$M - \varepsilon < x_1 \leqslant M.$$

Let the set E be bounded below.

A number m (finite) is called the *greatest lower bound* or *infimum* of a set E if the following two conditions are fulfilled for it:

(1) $m \leqslant x, \ \forall x \in E$,

(2) for any $\varepsilon > 0$ there exists a point $x_1 \in E$ such that

$$m \leqslant x_1 < m + \varepsilon.$$

Obviously, if there is a maximum (minimum) number in the set E of real numbers, i.e. if there exists $\max E$ ($\min E$), then

$$\sup E = \max E \quad (\inf E = \min E)$$

("sup" and "inf" are, respectively, the abbreviations of the Latin supremum meaning "the highest", "the greatest" and infimum "the lowest"). This terminology is not quite felicitous, since, for instance, $\sup E$ is not always the highest element in the set E.

Example 1. The set

$$E = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\} = \left\{\frac{n}{n+1}\right\}$$

has the least number equal to $1/2$ ($\min E = 1/2$). But it has no greatest number, since $\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \ldots$. Nevertheless, it is bounded above by the number 1 or

by any number greater than 1. But the number 1 plays an exceptional role, since it is the supremum of E ($\sup E = 1$).

Indeed:

(1) $\frac{n}{n+1} < 1, \quad \forall n \in \mathsf{N}$,

(2) $\forall \varepsilon > 0, \quad \exists n_1 \in \mathsf{N}\colon\ 1 - \varepsilon < \frac{n_1}{n_1+1} < 1.$

We have defined the supremum (infimum) for a set bounded above (below).

If E is not bounded above (below), then the symbol

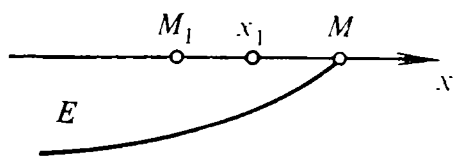

Fig. 9

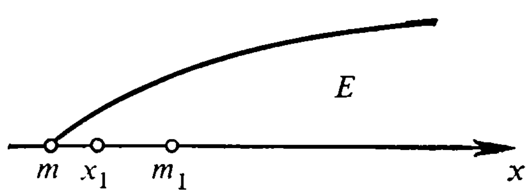

Fig. 10

$+\infty$ ($-\infty$) is said to be its supremum (infimum): $\sup E = +\infty$ (respectively, $\inf E = -\infty$).

Sometimes, when there is no danger of confusion, instead of $+\infty$ we write simply ∞.

Examples. For the above mentioned sets (1)-(6), we have

$$\sup \mathsf{Z} = +\infty, \qquad \inf \mathsf{Z} = -\infty,$$
$$\sup \mathsf{N} = \infty, \qquad \inf \mathsf{N} = \min \mathsf{N} = 1,$$
$$\sup \mathsf{N}_- = \max \mathsf{N}_- = -1, \quad \inf \mathsf{N}_- = -\infty,$$
$$\sup (a, b) = b, \qquad \inf (a, b) = a,$$

where a and b may be finite and infinite numbers.

It is possible to give a general definition of the supremum (infimum) of a set which will suit any set (bounded or unbounded).

A number M (m) is called the *supremum* (*infimum*) of a set E (Figs. 9 and 10) if the following conditions are fulfilled:

(1) $x \leqslant M$ $(m \leqslant x)$, $\forall x \in E$;

(2) for any (finite!) $M_1 < M$ $(m_1 > m)$ there exists $x_1 \in E$ such that $M_1 < x_1 \leqslant M$ $(m \leqslant x_1 < m_1)$.

We do not have to utilize the difference $M - \varepsilon$ (sum $m + \varepsilon$) in this formulation, since this has no sense for $M = +\infty$ $(m = -\infty)$.

The following theorem of principal importance is valid.

Theorem 1. *If a nonempty set E of real numbers is bounded above (below) by a finite number K (respectively, k), then there exists a number $M \leqslant K$ $(m \geqslant k)$, which is the supremum (infimum) of E.*

Proof. Since E is not an empty set, it contains at least one point x_0. Let us consider the closed interval $\sigma_0 = [a, b]$, where $a < x_0$, $b = K$.

By the hypothesis, no points belonging to E lie to the right of σ_0. Let us separate σ_0 into two equal parts (two subintervals) and denote by σ_1 the right-hand half containing at least one point belonging to E. This should be understood in the sense that if both halves contain points belonging to E, then σ_1 will be the right-hand half, and if only one of them contains the points of E, then just this half is denoted by σ_1.

Let us denote by x_1 some point from E belonging to σ_1. Hence, $x_1 \in \sigma_1$, but there are no points of E to the right of σ_1. We now divide σ_1 into two equal subintervals and denote by σ_2 the newly obtained rightmost subinterval containing at least one point of E which will be denoted by x_2. There are no points of E to the right of σ_2.

Continuing this process by induction, we get a sequence of nested intervals $\sigma_n = [a_n, b_n]$ $(\sigma_n \supset \sigma_{n+1})$, whose lengths

$$b_n - a_n = \frac{b-a}{2^n} \to 0, \quad n \to \infty.$$

In this case, for any $n \in \mathbf{N}$ there are no points of E to the right of σ_n, but σ_n contains a certain point $x_n \in E$.

According to the principle of nested intervals, there exists a unique point, which will be denoted by M, belonging to all the intervals σ_n $(M \in \sigma_n, \forall n)$.

Let us prove that

$$M = \sup E. \tag{1}$$

Indeed:

(1) there holds the inequality $x \leqslant M$, $\forall x \in E$, since if x is a certain point belonging to E, then $x \leqslant b_n$, $\forall n$, and, passing to the limit as $n \to \infty$, we obtain

$$x \leqslant M = \lim_{n\to\infty} b_n, \quad \forall x \in E;$$

(2) for any $\varepsilon > 0$, $\exists x' \in E$

$$M - \varepsilon < x' \leqslant M. \tag{2}$$

Indeed, the above defined points x_n belong respectively to σ_n and E, i.e. $a_n \leqslant x_n \leqslant M$, and since $a_n \to M$, $n \to \infty$, for any $\varepsilon > 0$, $\exists n_0$

$$M - \varepsilon < a_{n_0} \leqslant x_{n_0} \leqslant M,$$

and we have obtained (2), making $x' = x_{n_0}$.

The corresponding theorem which states that if a set E is bounded below then it has an infimum is proved in a similar way. In this case we proceed from the interval $\sigma_0 = [a, b]$ containing a certain point $x_0 \in E$, such that $a = k$ and $x_0 < b$. We divide σ_0 into two equal subintervals this time denoting by σ_1 the leftmost half containing the points belonging to E, find in σ_1 the point $x_1 \in E$, and continue then this process by induction.

The aforesaid leads us to the following assertion: *any set E has both a supremum and an infimum. If E is bounded above, then* $\sup E < \infty$, *and if E is not bounded above, then* $\sup E = \infty$. *Analogously, if E is bounded below, then* $\inf E > -\infty$, *and if E is not bounded below, then* $\inf E = -\infty$.

Problems

1. Let there be given two sets of real numbers $X = \{x\}$, $Y = \{y\}$. By the set $\{x + y\}$ we shall understand all possible sums of the numbers $x \in X$ and $y \in Y$. Prove that

$$\sup \{x + y\} = \sup \{x\} + \sup \{y\},$$

$$\inf \{x + y\} = \inf \{x\} + \inf \{y\}.$$

2. By the set $\{xy\}$ we shall understand all possible products of nonnegative numbers $x \in X$ and $y \in Y$. Prove that

$$\sup \{xy\} = \sup \{x\} \sup \{y\}, \quad \inf \{xy\} = \inf \{x\} \inf \{y\}$$
$$(x \geqslant 0,\ y \geqslant 0).$$

3. Prove that

$$\sup_{x\in A} (-x) = -\inf_{x\in A} x, \quad \inf_{x\in A} (-x) = -\sup_{x\in A} x.$$

Sec. 2.9. The Bolzano-Weierstrass Theorem *

Let there be given an arbitrary sequence of real numbers $\{x_n\}$. Let us choose from it an infinite set of terms numbered as follows: $n_1 < n_2 < \ldots$. Then we shall obtain a new sequence $\{x_{n_k}\}$ which is called the *subsequence of the sequence* $\{x_n\}$. We can separate infinitely many such subsequences from the given sequence.

If the sequence $\{x_n\}$ converges (to a finite number, $+\infty$ or $-\infty$), then it is obvious that any of its subsequences also converges, and also to the same number (finite, or equal to, $+\infty$ or $-\infty$).

The sequence

$$\{1, -1, 1, -1, \ldots\} \tag{1}$$

may serve as an example of a nonconvergent sequence of numbers. Nevertheless, we see that this sequence contains the subsequence

$$\{1, 1, 1, \ldots\},$$

which converges to 1. There arises a question whether this is always so, whether any sequence of real numbers contains a subsequence converging to a certain number (finite one, or equal to $+\infty$ or $-\infty$). The following theorem gives a positive answer to this question.

Theorem 1. *From any sequence of real numbers* $\{x_n\}$ *it is possible to single out a subsequence* $\{x_{n_k}\}$ *converging either to a finite number, or to* $+\infty$, *or to* $-\infty$.

If the sequence $\{x_n\}$ is not bounded above (below), then it, obviously, contains a subsequence tending to $+\infty$ (to $-\infty$) which proves the theorem. And if the sequence is bounded, then Theorem 1 is reduced to the following theorem.

Theorem 2. (Bolzano-Weierstrass theorem). *From any bounded sequence* $\{x_n\}$ *it is possible to separate a subsequence* $\{x_{n_k}\}$ *convergent to a certain number.*

---

* Bolzano, Bernhard (1781-1848). Czech mathematician.
Weierstrass, Karl Theodor Wilhelm (1815-1897). German mathematician.

Proof. Since the sequence of points $\{x_n\}$ is bounded, all of the points belong to a closed interval $[a, b]$ which will be denoted by σ_0. We then divide this interval into two equal parts and denote by σ_1 the rightmost of these parts containing an infinite number of terms x_n. Let us denote one of these terms by x_{n_1}. If there are points x_n to the right of σ_1, then their number is finite. Dividing the interval σ_1 into two equal parts, we denote by σ_2 the rightmost of the parts containing an infinite number of terms x_n. Among these terms there is, obviously, a term x_{n_2} with $n_2 > n_1$. If there are points x_n to the right of σ_2, then their number is finite.

Let us continue this process by induction. As a result, we shall obtain a sequence of nested intervals $\sigma_k = [a_k, b_k]$, whose lengths $b_k - a_k \to 0$, $k \to \infty$, and a subsequence of points belonging to our sequence such that $x_{n_k} \in \sigma_k$ $(n_1 < n_2 < \ldots)$. In this case, there are not more than a finite number of terms x_n to the right of each of the obtained intervals.

According to the principle of nested intervals, there exists a point c belonging to any of the intervals σ_k. Obviously, the subsequence $\{x_{n_k}\}$ has c as its limit $(x_{n_k} \to c)$, and we have thus proved the theorem.

Sec. 2.10. Limit Superior and Limit Inferior

If there is given an arbitrary sequence of real numbers $\{x_n\}$ then, according to Theorem 1 of the preceding section, it is possible to consider different convergent subsequences generated by it.

The *limit superior of a sequence* $\{x_n\}$ (or a *variable* x_n) is defined as a number M (finite, or equal to, $+\infty$ or to $-\infty$) possessing the following two properties:

(1) There exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ convergent to M:

$$\lim_{k\to\infty} x_{n_k} = M.$$

(2) For any convergent subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$

$$\lim_{k\to\infty} x_{n_k} \leqslant M.$$

The limit superior of the sequence $\{x_n\}$ is designated by one of the following symbols:

$$M = \overline{\lim}\, x_n = \overline{\lim_{n\to\infty}}\, x_n = \limsup_{n\to\infty} x_n.$$

If the sequence $\{x_n\}$ is not bounded above, then, obviously,

$$\overline{\lim}\, x_n = +\infty.$$

In the sequence $\{(-1)^n\}$ the variable x_n has $\overline{\lim}\, x_n = 1$.

Here is another example

$$\{n^{(-1)^n}\} = \left\{1,\ 2,\ \frac{1}{3},\ 4,\ \frac{1}{5},\ \ldots\right\}.$$

This sequence (variable) is not bounded above. Consequently, its limit superior

$$\overline{\lim}\, n^{(-1)^n} = +\infty.$$

For a bounded above sequence $\{x_n\}$ its limit superior M may also be defined in the following way: for any $\varepsilon > 0$ there are not more than a finite number of points x_n to the right of $M + \varepsilon$, and there are, automatically, infinitely many points to the right of $M - \varepsilon$.

Note that if a sequence $\{x_n\}$ has an ordinary (finite) limit $\lim x_n = M$, then, as we know, for any $\varepsilon > 0$ the inequalities $M - \varepsilon < x_n < M + \varepsilon$ are fulfilled for all x_n except their finite number. Thus, there are not more than a finite number of terms x_n to the right of $M + \varepsilon$, and there are, automatically, infinitely many terms x_n to the right of $M - \varepsilon$.

This shows that also $M = \overline{\lim}\, x_n$.

Hence, if $M = \lim x_n$, then also $\overline{\lim}\, x_n = \lim x_n = M$.

But the difference between the ordinary limit and the limit superior consists in the fact that in the case of the limit there are not more than a finite number of points x_n to the left of $M - \varepsilon$, whereas in the case of the limit superior there may be infinitely many points x_n as well.

The *limit inferior of a sequence* $\{x_n\}$ (or a *variable* x_n) is defined as a number m (finite or equal to $+\infty$ or to $-\infty$) possessing the following properties:

(1) There exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ convergent to m:

$$\lim_{k\to\infty} x_{n_k} = m.$$

(2) For any convergent subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$

$$\lim_{k\to\infty} x_{n_k} \geqslant m.$$

The limit inferior of the variable x_n is denoted by one of the following symbols:

$$m = \underline{\lim}\, x_n = \underline{\lim}_{n\to\infty}\, x_n = \liminf_{n\to\infty} x_n.$$

If the sequence $\{x_n\}$ is not bounded below, then, obviously,

$$\underline{\lim}\, x_n = -\infty.$$

For a bounded below sequence the limit inferior m can also be defined in the following manner: for any $\varepsilon > 0$ there are not more than a finite number of points (terms) x_n to the left of $m - \varepsilon$, and there are, automatically, infinitely many points (terms) to the left of $m + \varepsilon$.

It is obvious that

$$\underline{\lim}\, x_n \leqslant \overline{\lim}\, x_n. \tag{1}$$

Theorem 1. *For a sequence* $\{x_n\}$ *to have a limit (finite, or equal to* $+\infty$ *or to* $-\infty$*), it is necessary and sufficient that* $\overline{\lim}\, x_n = \underline{\lim}\, x_n$, *and then* $\lim x_n = \underline{\lim}\, x_n = \overline{\lim}\, x_n$.

Note that if $\overline{\lim}\, x_n = -\infty$, then by (1), $\underline{\lim}\, x_n = -\infty$, and, by Theorem 1,

$$\lim x_n = -\infty.$$

It is also obvious that from the equality $\underline{\lim}\, x_n = +\infty$ it follows that

$$\overline{\lim}\, x_n = \lim x_n = +\infty.$$

Remark. It can be shown that the number c obtained in proving the Bolzano-Weierstrass theorem is the limit superior of x_n:

$$\overline{\lim}\, x_n = c.$$

This follows from the fact that there are not more than a finite number of points x_n to the right of each interval σ_n.

On the other hand, if we modify the process by choosing at each step of dividing σ_n into two equal parts not the rightmost, but the leftmost of them containing an infinitude of points x_n, we would obtain, perhaps, another point c' contained in all σ_n, and this point would be the limit inferior of x_n ($\underline{\lim}\, x_n = c'$).

If the variable x_n has no limit, then, automatically, $c' < c$, and if the limit of x_n exists, then both processes will necessarily lead to one and the same number $c = c'$.

Sec. 2.11. Cauchy Condition for Convergence of a Sequence

Let there be given a sequence of real numbers $\{x_n\}$ converging to a finite limit a:

$$\lim_{n\to\infty} x_n = a.$$

This means that for any $\varepsilon > 0$ there exists a number $n_0 = n_0(\varepsilon)$ such that

$$|x_n - a| < \varepsilon/2, \quad \forall n > n_0.$$

Instead of the natural number $n > n_0$, we may substitute another natural number $m > n_0$ into this inequality:

$$| x_m - a | < \varepsilon/2, \qquad \forall m > n_0.$$

Then

$$|x_n - x_m| = |x_n - a + a - x_m| \leqslant |x_n - a| + |x_m - a|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n, \ m > n_0.$$

We have obtained the following assertion: *If a variable x_n has a finite limit, then the following condition* (Cauchy's*) *is fulfilled for it: for any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that*

$$| x_m - x_n | < \varepsilon, \quad \forall n, \ m > n_0.$$

A sequence of numbers satisfying the Cauchy condition is also called a *fundamental sequence.*

It turns out that the converse is also true: *If a sequence of real numbers $\{x_n\}$ is fundamental, i.e. satisfies the Cauchy condition, then it has a limit, that is, there exists a number a (finite) such that $x_n \to a$, $n \to \infty$.*

Proof. To begin with, let us prove that a fundamental sequence is bounded. Indeed, let us set $\varepsilon = 1$ and choose, according to the Cauchy condition, the number $n_0 = n_0(1)$ such that

$$| x_n - x_m | < 1, \quad \forall n, \ m > n_0,$$

whence

$$1 > | x_n - x_m | \geqslant | x_n | - | x_m |$$

or

$$1 + | x_m | \geqslant | x_n |, \quad \forall n, \ m > n_0. \tag{1}$$

Let us have fixed $m > n_0$ and denote

$$M = \max_{n \leqslant n_0} \{1 + | x_m |, | x_n |\},$$

i.e. the maximum of the numbers $| x_n |$, where $n \leqslant n_0$, and the numbers $1 + | x_m |$. Then, by (1), we have

$$M \geqslant | x_n |, \quad \forall \, n \in \mathbf{N},$$

and the boundedness of the sequence $\{x_n\}$ has been thus proved.

According to the Bolzano-Weierstrass theorem, from a bounded

* Cauchy, Augustin Louis (1789-1857). Famous French mathematician. He was the first to state the definitions of the basic notions of mathematical analysis (limit, continuity, integral, etc.) in the way characteristic of modern mathematics.

sequence $\{x_n\}$ it is possible to separate a subsequence $\{x_{n_k}\}$ converging to a certain (finite) number a, i.e.

$$\lim_{k\to\infty} x_{n_k} = a.$$

Let us show that in this case not only this subsequence, but also the entire sequence has the limit a:

$$\lim_{n\to\infty} x_n = a.$$

Indeed, according to Cauchy's condition, which is satisfied by our sequence, for any $\varepsilon > 0$, we can find an n_0 such that

$$|x_n - x_m| < \varepsilon/2, \quad \forall n,\ m > n_0. \tag{2}$$

On the other hand, since $x_{n_k} \to a$, $k \to \infty$, we can indicate k_0 such that

$$|x_{n_k} - a| < \varepsilon/2, \quad \forall k > k_0.$$

Taking into consideration that $n_k \to \infty$ as $k \to \infty$, we can find $k_1 > k_0$ such that $n_{k_1} > n_0$. Therefore

$$|x_{n_{k_1}} - a| < \varepsilon/2. \tag{3}$$

By (2), where it is necessary to set $m = n_{k_1}$, and (3), we have

$$|x_n - a| = |x_n - x_{n_{k_1}} + x_{n_{k_1}} - a| \leqslant |x_n - x_{n_{k_1}}| + |x_{n_{k_1}} - a|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n > n_0,$$

and we have thus proved that the sequence $\{x_n\}$ has a limit equal to a.

Hence, the following theorem has been proved:

Theorem 1 (Cauchy's criterion for existence of a limit). *For a sequence of real numbers $\{x_n\}$ to have a limit, it is necessary and sufficient that it should be fundamental (should satisfy Cauchy's condition).*

Sec. 2.12. Completeness and Continuity of the Set of Real Numbers

In the previous sections we proved a number of properties of real numbers the most important of which are listed below:

(1) Existence of a limit of a bounded monotonic sequence (Sec. 2.5, Theorem 1).

(2) The principle of nested intervals (Sec. 2.7, Theorem 1).

(3) Existence of the supremum of an arbitrary bounded set (Sec. 2.8, Theorem 1).

(4) Convergence of a fundamental sequence to a limit (Cauchy's criterion, Sec. 2.11, Theorem 1).

Although these properties look differently, in fact there is a profound internal relationship among them. It is not difficult to show that assertions (1)-(4) (in the presence of properties I-IV of real numbers) are equivalent to one another, i.e. any of them yields the three remaining assertions. As it was shown above, (2), (3), and (4) follow from (1) (or, which is the same, from Property V, see Sec. 1.6) and properties I-IV.

Properties (1)-(4) are also called the *properties of continuity* or *completeness* of the set of all real numbers.

In order to understand their role, let us consider the set of only rational numbers which will be denoted by Q.

Properties I-IV are fulfilled for rational numbers. But Property V and, consequently, any of properties (1) to (4) are, generally speaking, not fulfilled for them.

Let us clarify this by a particular example. To this end, it will be convenient for us to operate also with the set of all real numbers which will be denoted by R.

Let there be given an infinite nonrepeating decimal

$$a = a_0 \cdot a_1 a_2 a_3 \ldots.$$

Hence, a is an irrational number, i.e. $a \in \mathsf{R}$, but $a \bar{\in} \mathsf{Q}$. The decimal a generates a sequence of truncated rational numbers

$$a^{(n)} = a_0 \cdot a_1 \ldots a_n \quad (n = 1, 2, \ldots).$$

This is a nondecreasing sequence bounded above by an integer $a_0 + 1$. There is no rational number to which our sequence of rational numbers $\{a^{(n)}\}$ converges. Indeed, we know that the variable $a^{(n)}$ converges to a (see Example 9, Sec. 2.1), i.e. to an irrational number, and it cannot converge to another number.

We have shown that Property (1) is, generally speaking, not fulfilled in Q.

It can be easily shown that properties (2), (3), and (4) are also, generally speaking, not fulfilled in Q.

The set of real numbers is called *complete* due to the fact that for it there is fulfilled Property (4) consisting in that any fundamental sequence of numbers is convergent to some real number. The set Q of rational numbers is not complete. It contains fundamental sequences not converging to rational numbers. By adding irrational numbers to Q, we complete the space of real numbers.

CHAPTER 3

FUNCTION. LIMIT OF FUNCTION

Sec. 3.1. Function

Let E be a set of numbers and let there be given a certain law according to which with every number x belonging to E a (single) number y is associated. Then we say that there is a (single-valued or one-valued) *function* defined on E. In this case we write

$$y = f(x) \quad (x \in E). \tag{1}$$

This definition of a function was suggested by N. I. Lobachevsky and P. G. L. Dirichlet*. The set E is called the domain of definition (or, simply, the domain) of the function $f(x)$. We also say that we are given an *independent variable* x which can assume the particular values of x belonging to the set E and that, according to the given law, to each $x \in E$ there corresponds a certain value (number) of the other variable y called a *function* or a *dependent variable*. The independent variable is also called the *argument*.

The concept of a function can be interpreted in geometrical terms. We say that there is a set E of points x of the real line called, as was said, the *domain of definition of the function* and a law associating with every point $x \in E$ a number $y = f(x)$.

If we speak of a function as of a certain law of correspondence associating with each number $x \in E$ a number y, then it is sufficient to denote the function by a single letter f. The symbol $f(x)$ denotes the number y which, in accordance with the law f, corresponds to the value $x \in E$. For instance, if the number 1 belongs to the

---

* Lobachevsky, Nikolai Ivanovich (1792-1856). Great Russian mathematician, the creator of the non-Euclidean geometry.

Dirichlet, Peter Gustav Lejeune (1805-1859). German mathematician.

domain of definition E of a function f, then $f(1)$ is the *particular value of the function* f at the point $x = 1$. If 1 does not belong to E (i.e. $1 \bar{\in} E$), then we say that the function f is *not defined* at the point $x = 1$.

The set E_1 of all the values $y = f(x)$, where $x \in E$ is called the *image* of the set E (produced by the function f) or the range of the function $y = f(x)$ for $x \in E$. In such a case we sometimes write $E_1 = f(E)$. In using the latter notation care should be taken not to confuse it with the notation $y = f(x)$, where x is an arbitrary number (point) belonging to the set E and y is the corresponding point of the set E_1 assigned to x by the function (law) f; to this end it is advisable to make the necessary stipulation each time the symbolic relation $E_1 = f(E)$ is used. We also say that the function f *maps* the set E *on* the set E_1 and sometimes synonymously call f a mapping.

If $E_1 = f(E) \subset A$, where A is a set of numbers which, in the general case, may not coincide with E_1, then we say that f *maps* E into A.

For two functions f and φ defined on one and the same set E we define the *sum* $f + \varphi$, the *difference* $f - \varphi$, the *product* $f\varphi$, and the *quotient* f/φ. These are new functions expressed respectively by the formulas

$$f(x) + \varphi(x), \quad f(x) - \varphi(x), \quad f(x)\,\varphi(x), \quad \frac{f(x)}{\varphi(x)} \quad (x \in E), \tag{2}$$

where in the case of the quotient it is supposed that $\varphi(x) \neq 0$ on E.

Any other letters F, Φ, Ψ, . . ., can also be used to denote functions, and instead of x and y we can write z, u, v, ...

If a function f maps a set E into E_1 and a function F maps the set E_1 into a set E_2, then the function $z = F(f(x))$ is called a *composite function*, or a *function of a function*, or the *superposition of the functions* f and F. This new function is defined on the set E and maps E into E_2.

A composite function can also be constructed as a superposition of any number n of functions: $z = F_1(F_2(F_3(\ldots(F_n(x))\ldots)))$.

In practice we encounter many examples of functions. For instance, the area S of a circle of radius r is a function of the radius r expressed by the formula $S = \pi r^2$. This function is obviously defined on the set of all positive numbers r.

The dependence of a variable S on a variable r expressed by the formula $S = \pi r^2$ can be considered irrespective of the problem of computing the area of a circle. The function $S = \varphi(r)$ specified by this formula is defined throughout the entire real axis, that is for all real numbers r, not necessarily for positive ones.

Given below are examples of functions determined by formulas:

$$(1)\ y=\sqrt{1-x^2},\quad (2)\ y=\log(1+x),\quad (3)\ y=x-1,$$

$$(4)\ y=\frac{x^2-1}{x-1},\quad (5)\ y=\arcsin x.$$

Here we mean real functions assuming real values y for the corresponding real values of the argument x. It is easily seen that the domains of definition of these functions are, respectively,

(1) the closed interval $[-1, 1] = \{-1 \leqslant x \leqslant 1\}$;
(2) the set $x > -1$;
(3) the entire real axis;
(4) the entire real axis with the point $x = 1$ deleted;
(5) the closed interval $[-1, +1]$.

The functions in examples (1) and (2) can be regarded as composite functions: (1) $y = \sqrt{u}$, $u = 1 - v$, $v = x^2$; (2) $y = \log u$, $u = 1 + x$.

The construction of the graphs of functions is an important method of representing functions. Let us take a rectangular coordinate system x, y (Fig. 11). Marking a closed interval $[a, b]$ on the x-axis, we plot an arbitrary curve Γ possessing the following property: for any point $x \in [a, b]$ the straight line parallel to the y-axis and passing through this point meets the curve Γ at exactly one point A. A curve of this kind specified in rectangular (Cartesian) coordinate system will be referred to as a *graph*. The given graph determines a function $y = f(x)$ on the interval $[a, b]$: if x is an arbitrary point

of the interval $[a, b]$, then the corresponding value $y = f(x)$ of the function is determined as the ordinate of the point A (see Fig. 11). Hence, by means of the graph we have obtained a definite law of correspondence between x and $y = f(x)$.

We have defined, by means of a graph, a function on the set E which is a closed interval $[a, b]$. In other cases E can be an open interval, a half-interval, the entire real axis, the set of rational points belonging to a given interval and the like.

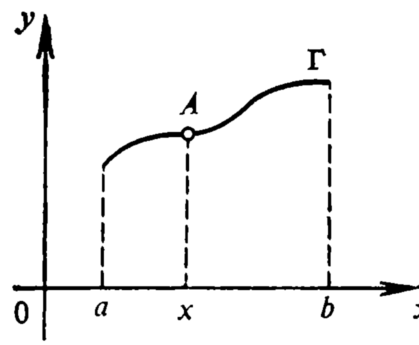

Fig. 11

Let us take a function $f(x)$ defined on an open interval (a, b) and an arbitrary (constant) number $\alpha \neq 0$. Using α and f, we can construct a number of other functions: (1) $\alpha f(x)$; (2) $f(x) + \alpha$; (3) $f(x - \alpha)$; (4) $f(\alpha x)$. Functions (1) and (2) are defined on the same interval (a, b). The ordinates of the graph of function (1) are α times the corresponding ordinates of the function $f(x)$. The graph of function (2) is obtained from that of f by moving the latter graph α units of length upward if $\alpha > 0$, and $|\alpha|$ units downward if $\alpha < 0$. As to the graph of function (3), it is obtained from the graph of f by shifting the latter by a distance of α to the right if $\alpha > 0$, and by the distance of $|\alpha|$ to the left if $\alpha < 0$. Finally, function (4) is obviously defined, for $\alpha > 0$, on the interval $(a/\alpha, b/\alpha)$; its graph being obtained by an α-fold uniform contraction of the graph of f. A function f is used to be *even* or *odd* if it is defined on a set symmetric with respect to the origin and if it possesses, respectively, the property $f(-x) = f(x)$ or $f(-x) = -f(x)$.

The graph of an even function is apparently symmetric about the y-axis, while the graph of an odd function is symmetric about the origin. For instance, x^{2k} (where k is a natural number), $\cos x$, $\log|x|$, $\sqrt{1 + x^2}$, $f(|x|)$ are even functions, while x^{2k+1} (where $k \geqslant 0$ is an integer), $\sin x$, $x\sqrt{1 + x^2}$, $xf(|kx|)$ are odd ones.

It is easily seen that the *product of two even or of two odd functions is an even function and the product of an even function by an odd function is an odd function.*

Most of the functions are, of course, neither even nor odd.

The graph of a function $y = f(x)$, $x \in E$, can also be defined as the set of points $(x, f(x))$ with the abscissa x and ordinate $f(x)$, where $x \in E$.

A function f is called *increasing* (*nondecreasing*) on E if for any $x_1, x_2 \in E$, for which $x_1 < x_2$, the inequality $f(x_1) < f(x_2)$ $(f(x_1) \leqslant f(x_2))$ is fulfilled.

A function f is called *decreasing* (*nonincreasing*) on E if for any $x_1, x_2 \in E$, for which $x_1 < x_2$, the inequality $f(x_1) > f(x_2)$ $(f(x_1) \geqslant f(x_2))$ is fulfilled.

A function f is called *bounded* (*unbounded*) on E if its image $E_1 = f(E)$ with the aid of f is a bounded (unbounded) set.

For example, the function $y = 1/x$ decreases and is not bounded on $(0, \infty)$, but is bounded on $[1, \infty)$.

A function f defined on the entire real axis is called *periodic with the period* $T > 0$ if $f(x) = f(x + T)$ for any x.

We can also speak of a function periodic (with the period T) on an open interval (a, b) (on a closed interval $[a, b]$) if the equality

$$f(x) = f(x + T)$$

is true for all such $x \in (a, b)$ (or $[a, b]$), for which $x + T \in (a, b)$ ($[a, b]$).

For instance, a periodic function $\sin x$ has a period 2π. The function $\sin mx$, where $m \in \mathbf{N}$, is also of period 2π, but it has a smaller period $T = 2\pi/m$.

Example 6. The *signum* function

$$y = \operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases}$$

is defined on an infinite interval $(-\infty, \infty)$. It is odd. Its image is a set consisting of three points: 1, 0, and -1.

Example 7. The function

$$y=\begin{cases} x^2+1, & x\leqslant 0,\\ \sin x, & x>0,\end{cases}$$

has a graph represented in Fig. 12. It decreases on $(-\infty, 0)$ and has a period 2π on $(0, \infty)$. This function is specified by different formulas on different parts of its domain of definition.

A function can be specified by a table. For instance, if we measure the air temperature T every hour during

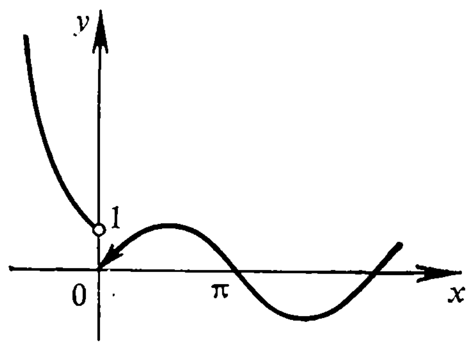

Fig. 12

twenty-four hours, then to each time moment $t = 0, 1, 2, \ldots, 24$ there corresponds a definite number T, the variation of T being determined by a table of the form

t	0	1	...	24
T	T_0	T_1	...	T_{24}

We thus obtain a function $T = f(t)$, defined on the set of integers from 0 to 24 represented by this table.

If a function $y = f(x)$ is defined on a set E by means of a formula, then we can always speak of a definite graph specifying this function geometrically. The converse is by far not always so clear: the question as to whether

a function specified by an arbitrary graph can be expressed by a formula is rather difficult. To answer it we have to define more precisely what is meant by the word "formula". Above, when we spoke of a given function $y = f(x)$ determined by a formula, we supposed tacitly that y can be obtained from x by a finite number of such operations as addition, subtraction, multiplication, division, extraction of the kth root with a certain k, taking logarithms, performing the operations symbolized by sin, cos, arc sin, and other algebraic and trigonometric operations.

Mathematical analysis provides certain means for an essential generalization of the notion of a formula. An important operation of that kind is the expansion of a function in an infinite series with respect to elementary functions.

Many, if not all, functions encountered in practice can be represented by a formula of the type of an infinite series whose terms are elementary functions defined below; the notion of a series will be defined later on and now we are not going to speak of it, since we are not yet ready to discuss this question.

In any event, irrespective of the way in which a function $f(x)$ is specified, by a formula, or a graph, or in some other way, it can serve as the object to which the methods of mathematical analysis can be applied if it satisfies some additional general requirements, such as continuity, monotonicity, convexity, differentiability, and the like. But this is to be discussed below.

An important notion which is used in the investigation of functions is that of limit; the notion of a limit is one of the most important in mathematical analysis. The present chapter is devoted to the study of this notion.

If to each number x belonging to a given set E of numbers there corresponds, in accordance with a certain law, a definite set e_x of numbers y, then we say that this law of correspondence determines a *many-valued (multiple-valued) function* $y = f(x)$. If it turns out that for every $x \in E$ the set e_x consists only of a single number y, then we obtain, as a special case, a *single-valued (one-valued)* function,

A one-valued function is frequently spoken of as a "function" without mentioning "one-valued" when this does not lead to a misunderstanding.

In algebra and trigonometry we often deal with multiple-valued functions; such are, for instance, $\sqrt{x}$, Arc sin x, Arc tan x, etc.

The function $\sqrt{x}$ is defined for $x \geqslant 0$. It is two-valued, since to every positive number x there correspond two real numbers differing in sign whose squares are equal to x. However, the symbol $\sqrt[k]{x}$ $(k = 2, 3, \ldots)$ will always be understood, unless otherwise stated, as the arithmetic kth root of $x \geqslant 0$, that is, as a nonnegative number whose kth power equals x (see Sec. 3.8). As far as the function Arc sin x is concerned, it is infinite-valued. In the functional relationship established by the latter function to each value of x belonging to the closed interval $[-1, 1]$ there corresponds an infinite set of numbers y which can be written by means of the formula

$$y = (-1)^k \arc \sin x + k\pi \quad (k = 0, \pm 1, \pm 2, \ldots).$$

Our discussion concerned functions of one variable, but we can also speak of functions of two, three, and, generally, n variables.

The notion of a function of two variables is introduced in the following way. We consider a set E consisting of pairs of numbers. Here we mean *ordered* pairs: two pairs (x_1, y_1) and (x_2, y_2) are regarded as equal (coinciding) if and only if $x_1 = x_2$ and $y_1 = y_2$. If there is a certain law which assigns to every pair $(x, y) \in E$ a number z, then we say that this correspondence specifies a *function* $z = f(x, y)$ *of the two variables* x and y on the set E.

Since to every pair of numbers (x, y) there corresponds a definite point with coordinates x and y, in the plane where the Cartesian coordinates x and y are introduced and, conversely, to each point there corresponds a pair (x, y), we can say that the function $z = f(x, y)$ is defined on a set E of points lying in the plane.

Taking a three-dimensional space with rectangular coordinates x, y, and z, we can represent a function

$z = f(x, y)$ of two variables by the locus of points $(x, y, f(x, y))$ whose projections (x, y) on the coordinate plane xy belong to the domain of definition E of the function f.

For example, for the function

$$z = \sqrt{1 - x^2 - y^2}, \quad x^2 + y^2 \leqslant 1,$$

the corresponding locus is the upper hemisphere of radius 1 with centre at the origin.

A function of three independent variables can be defined in a similar way. In this case the domain of definition is a set E consisting of ordered triples of numbers (x, y, z) or, which is the same, the set of the corresponding points of a three-dimensional space where a Cartesian coordinate system is introduced.

If to every triple of numbers (i.e. to every point of the three-dimensional space) (x, y, z) belonging to E there corresponds, in accordance with a certain law, a number u, then we say that this correspondence determines a function $u = F(x, y, z)$ defined on the set E.

Similarly, we can consider a set E of ordered n-tuples $(x_1, \ldots, x_n)$ consisting of n numbers where n is a given natural number. Again, if to each such n-tuple belonging to E there corresponds, according to a certain law, a number z, then we say that z is a *function of the variables* $x_1, \ldots, x_n$ defined on the set E and we write $z = F(x_1, \ldots, x_n)$.

In the case $n > 3$ we no longer have at our disposal a real (geometrical) n-dimensional space which can be used for the visual representation of the n-tuples $(x_1, \ldots, x_n)$ as points belonging to that space. But mathematicians invented the concept of an n-dimensional space which successfully serves this purpose and not worse than the real geometrical three-dimensional space. Namely, by an *n-dimensional space* is meant the collection of all n-tuples.

If two functions f and φ of n variables are defined on one and the same set E of n-tuples $(x_1, \ldots, x_n)$ (which are regarded as points of the n-dimensional space), then the sum $f + \varphi$, the difference $f - \varphi$, the product $f\varphi$, and the quotient f/φ are specified as functions defined

on E, by means of equalities analogous to (2), the only distinction being that the numbers x should be replaced by the n-tuples $(x_1, \ldots, x_n)$. Composite functions such as $f(\varphi(x, y), \psi(x, y, z)) = F(x, y, z)$, where (x, y, z) are triples of numbers belonging to a certain set of triples are also defined in a natural manner in the case of three and, generally, n variables.

Given below are several examples of functions of many variables specified by means of elementary formulas.

Example 8. The expression $u = Ax + By + Cz + D$, where A, B, C, and D are given constant real numbers, is a linear function of three variables x, y, and z. It is defined throughout the three-dimensional space. A more general linear function of n variables $(x_1, \ldots, x_n)$ is specified by the formula $u = \sum_{i=1}^{n} a_i x_i + b$, where $a_1, \ldots, a_n$, b are given constant numbers. This function is defined at any point $(x_1, \ldots, x_n)$ of the n-dimensional space or, as we say, throughout the n-dimensional space.

Example 9. Consider the formula $z = \log \sqrt{1 - x^2 - y^2}$. This real function is defined in the domain representing a circle of radius 1 with centre at (0, 0) with all the boundary points, which are the points of the circumference of the circle, deleted. For the latter points our function is not defined, since log 0 makes no sense.

Example 10. The function

$$z = f(x, y) = \begin{cases} 0 & \text{for } y \geqslant 0, \\ 1 & \text{for } y < 0 \end{cases}$$

is represented geometrically by two disjoint parallel half-planes, their disposition relative to the coordinate system (x, y, z) being quite obvious.

A function of one variable can be determined *implicitly* by an equality

$$F(x, y) = 0 \tag{3}$$

where F is a function of the two variables x and y.

Let a function F be defined on a set G of points (x, y). Equality (3) specifies a subset Ω of the set G on which the function F is equal to zero. Of course, the subset Ω can be empty. Suppose that Ω is a nonempty set, and let E be a set (obviously, nonempty) of those values (numbers) x to each of which there corresponds at least one y such that the pair x, y belongs to Ω. The set E

thus consists of all numbers x to each of which there corresponds a nonempty set e_x of numbers y such that $(x, y) \in \Omega$ or, which is the same, such that for the pair (x, y) equality (3) is fulfilled. This determines on the set E a function $y = \varphi(x)$ of x which, in the general case, is many-valued. In such case we say that the function φ is *defined implicitly* by means of equality (3). It, obviously, satisfies the identity

$$F(x, \varphi(x)) \equiv 0 \quad \text{for all} \quad x \in E.$$

By analogy with the above, we can also define a function $x = \psi(y)$ of the variable y specified implicitly by equality (3). For this function there holds the identity

$$F(\psi(y), y) \equiv 0 \quad \text{for all} \quad y \in E_1,$$

where E_1 is a set of numbers y. We also say that the function $y = \varphi(x)$ (or $x = \psi(y)$) satisfies equation (3). The function $x = \psi(y)$ is called the *inverse* of the function $y = \varphi(x)$.

Example 11. The equation

$$x^2 + y^2 = r^2, \tag{4}$$

where $r > 0$ determines implicitly the two-valued function of one variable

$$y = \pm\sqrt{r^2 - x^2} \quad (-r \leqslant x \leqslant r)$$

which, by the way, is one-valued for $x = \pm r$. It is natural to regard this two-valued function as splitting into two continuous one-valued functions $y = +\sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$ $(-r \leqslant x \leqslant r)$. The locus formed of their graphs (which are the upper and the lower semicircles) is the circle of radius r with centre at the origin. This circle is the locus of points whose coordinates x and y satisfy equation (4). But it is possible, using formula (4), to construct various single-valued (discontinuous) functions satisfying equation (4). The following is an example of such functions:

$$y = \begin{cases} +\sqrt{r^2 - x^2}, & -r \leqslant x < 0, \\ -\sqrt{r^2 - x^2}, & 0 \leqslant x \leqslant r. \end{cases}$$

Sec. 3.2. Limit of a Function

A number A is called the *limit of a function f at a point a* if the function is defined in a neighbourhood of the point a, i.e. on an interval (c, d), where $c < a < d$, except possibly at the point a itself, and if for any $\varepsilon > 0$ there exists $\delta > 0$ (depending on ε) such that the inequality

$$|f(x) - A| < \varepsilon$$

holds for all x's for which $0 < |x - a| < \delta$.

The fact that A is the limit of f at a point a is customarily written as

$$\lim_{x \to a} f(x) = A \quad \text{or} \quad f(x) \to A \quad (x \to a).$$

An alternative definition of the limit of a function at a point can be formulated in terms of limits of sequences.

A number A is called the *limit of a function f at a point a* if the function is defined in a neighbourhood of the point a, except possibly at the point a itself, and if the limit of the sequence $\{f(x_n)\}$ exists and is equal to A for any sequence $\{x_n\}$ convergent to a such that $x_n \neq a$ for all n's. Thus,

$$\lim_{\substack{x_n \to a \\ x_n \neq a}} f(x_n) = A.$$

Here, as in all similar cases, it is tacitly implied that the variable x_n tending to a runs through a set of values for which $f(x)$ is defined.

These two definitions are equivalent. Indeed, let a function f have a limit in the sense of the first definition and let there be given a variable x_n tending to the number a and whose values do not coincide with a for any n. Setting an arbitrary ε, we choose δ according to the first definition. We then choose a natural n_0 such that $|x_n - a| < \delta$ for all $n > n_0$. But then

$$|f(x_n) - A| < \varepsilon \quad \text{for} \quad n > n_0,$$

and this means that the sequence of numbers $\{f(x_n)\}$ tends to A. Since this property holds for any sequence $\{x_n\}$ convergent to a and such that $x_n \neq a$ and all x_n

belong to the domain of definition of the function, we have thus proved that the second definition follows from the first definition.

Conversely, let a function $f(x)$ have a limit in the sense of the second definition. Suppose that it has no limit in the sense of the first definition. This means that there exists at least one value of ε, say ε_0, for which there is no δ mentioned in the first definition and that for *any* δ there is a value $x = x^{(\delta)}$ belonging to the set of numbers x satisfying the condition $0 < |x - a| < \delta$, such that $|f(x^{(\delta)}) - A| \geqslant \varepsilon_0$.

Let us consecutively take as δ all the numbers $\delta = 1/k$ $(k = 1, 2, \ldots)$ and find for each of them a point $x_k = x^{(\delta)}$ for which

$$0 < |x_k - a| < 1/k \quad (x_k \neq a)$$

and

$$|f(x_k) - A| \geqslant \varepsilon_0 \quad (k = 1, 2, \ldots).$$

These relations show that $x_k \to a$ $(x_k \neq a)$, while $f(x_k)$ does not automatically tend to the number A. Hence, the assumption that the second definition does not imply the first one leads to a contradiction.

We have thus proved the equivalence of the two definitions.

The expression "the limit of a function at a point a" is frequently replaced by the expression "the limit of a function for x tending to a" or, briefly, "the limit of a function as $x \to a$". The last two expressions are in a certain sense closer to the idea of the limit, since the expression $\lim\limits_{x \to a} f(x)$ provides some information on the behaviour of the function in a small neighbourhood of the point a from which this point is deleted. It says that when x attains values different from a and approaches a according to an arbitrary law, the corresponding value $f(x)$, in its turn, approaches A, i.e. becomes arbitrarily close to A.

Example 1. Let us consider the function $f(x) = (x^2 - 4)/(x - 2)$. It is defined for all $x \neq 2$. Let us try to find its limit as $x \to 2$. For any $x \neq 2$ we have

$\frac{x^2-4}{x-2} = x+2$, and since the definition of the limit as $x \to 2$ does not involve the value of the function f at the point $x = 2$, we get

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} (x+2).$$

The meaning of this equality is that if one of the limits entering into it exists, then the other also exists and is equal to the former. Hence, instead of evaluating the limit of a more complex function $(x^2-4)/(x-2)$, it is sufficient to find the limit of the simpler function $x+2$ as $x \to 2$. The latter limit is, obviously, equal to 4. Indeed, if we substitute any variable x_n tending to 2 for x into $x+2$, then, irrespective of the way in which x_n tends to 2, we obtain

$$\lim_{x_n\to 2} (x_n+2) = 2+2 = 4.$$

The calculations connected with the evaluation of the given limit are usually written in the following way:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} (x+2) = \lim_{x\to 2} x+2 = 4.$$

It should be stressed that $f(x) = (x^2-4)/(x-2)$ and $\varphi(x) = x+2$ are different functions. The former is defined for $x \neq 2$, while the latter for all x's. But when evaluating the limit of the functions as $x \to 2$, we are not interested in whether or not the functions are defined at the point $x = 2$ itself, and, since $f(x) = \varphi(x)$ for $x \neq 2$, we have

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \varphi(x) = \varphi(2).$$

Example 2. It is evident that $\lim\limits_{x\to 1} x^2 = 1$, since if $x_n \to 1$, $x_n \neq 1$, then $\lim x_n^2 = \lim x_n \cdot \lim x_n = 1\cdot 1 = 1$. This fact can also be proved in terms of ε and δ. Let us determine an interval containing the point 1, for instance (1/2, 3/2). For any x belonging to this interval the fol-

lowing inequality is, obviously, fulfilled

$$|x^2-1|=|x+1|\,|x-1|\leqslant\frac{5}{2}\,|x-1|.$$

Now, taking an arbitrary $\varepsilon>0$ and setting $\delta = \min\left\{\frac{1}{2}, \frac{2}{5}\,\varepsilon\right\}$, we obtain the relation

$$|x^2-1|\leqslant\frac{5}{2}\cdot\frac{2}{5}\,\varepsilon=\varepsilon,$$

for all x's satisfying the inequality $|x-1|<\delta$.

Example 3. The function $\sin(1/x)$ is defined for all $x\neq 0$ and is an odd one (its graph for $x>0$ is shown in Fig. 13). Hence, it is defined in any neighbourhood of the point $x=0$ except at the point $x=0$ itself. This

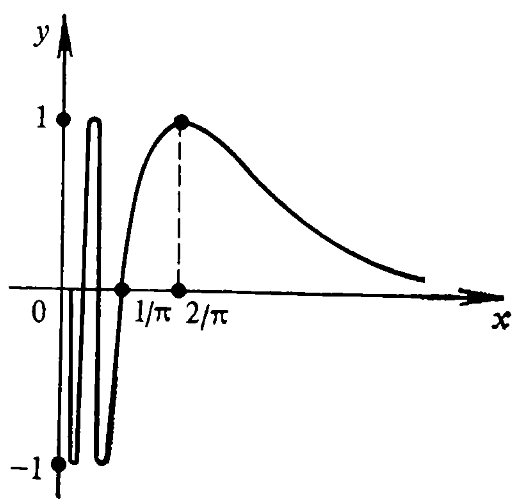

Fig. 13

function has no limit as $x\to 0$, since the sequence of values $x_k=2/\pi\,(2k+1)$ $(k=0, 1, 2, \ldots)$ which are different from zero, tends to zero, while the values

$$f(x_k)=(-1)^k$$

do not tend to any limit as $k\to\infty$.

Let us also introduce the following definition. We shall write

$$A=\lim_{x\to\infty} f(x)$$

and say that a *number A is the limit of the given function $f(x)$ for x tending to infinity* if f is defined for all x satisfy ing the inequality $|x| > K$ with some $K > 0$ and if for any $\varepsilon > 0$ we can find a number $M > K$ such that $|f(x) - A| < \varepsilon$ for all x satisfying the inequality $|x| > M$.

It can be proved that this definition is equivalent to the following one.

A number A is the limit of a function $f(x)$ for $x \to \infty$ if the function $f(x)$ is defined for all x's with $|x| > M$ for some M and if

$$\lim_{x_n \to \infty} f(x_n) = A$$

for any sequence $\{x_n\}$ converging to ∞.

The proof of the equivalence of these two definitions is carried out according to the scheme used above in the case of the limit of f at a finite point a.

Generally, many properties of limits of functions for $x \to a$, where a is a finite number, and for $x \to \infty$ are analogous. These properties admit of a unified presentation that simultaneously covers the cases when $x \to a$, where a is a finite number and when $x \to \infty$. To this end we shall agree to interpret the symbol a as any (finite) number or ∞. If a is a number, then by a neighbourhood of the point a we mean any interval (c, d) containing the point a. Hence, a *neighbourhood of a (finite) point a* is the set of all the points x satisfying the inequalities $c < x < d$. And if $a = \infty$ (or $+\infty$, or $-\infty$), then we shall agree to understand a neighbourhood of a as the set of all x satisfying the inequalities

$$|x| > M \quad (\text{or } x > M, \quad \text{or } x < -M, \quad M > 0).$$

We shall write

$$\lim_{x \to a} f(x) = A,$$

where a can be a finite number or ∞ (or $+\infty$, or $-\infty$) if the function $f(x)$ is defined in a neighbourhood of a except possibly at the point a itself (this stipulation is necessary only when a is a finite point), and if for any $\varepsilon > 0$ there is a neighbourhood of the point a such that

the inequality

$$|f(x) - A| < \varepsilon$$

is fulfilled for all x belonging to this neighbourhood and different from a.

The last definition of the limit f unifies all the cases treated above, i.e. the cases when x tends to a finite number a and when x tends to ∞ or $+\infty$, or $-\infty$.

A function f for which $\lim\limits_{x\to a} f(x) = 0$, is called *infinitesimal* as $x \to a$.

We now proceed to the study of the properties of a function $f(x)$ having limits as $x \to a$, where a is a (finite) number or ∞, or $+\infty$, or $-\infty$. Let us agree that an arbitrary neighbourhood of a will be denoted by the symbol $U(a)$. It is readily checked that the *intersection of two neighbourhoods* $U_1(a)$ and $U_2(a)$ *is again a neighbourhood* $U(a)$.

Theorem 1. *If* $\lim\limits_{x\to a} f(x) = A$, *where* A *is a finite number, then the function* $f(x)$ *is bounded in some neighbourhood* $U(a)$, *i.e. there exists a positive number* M *such that*

$$|f(x)| \leqslant M \text{ for all } x \in U(a), \quad x \neq a.$$

Proof. The conditions of the theorem guarantee the existence of a neighbourhood $U(a)$ such that

$$1 > |f(x) - A| \geqslant |f(x)| - |A| \quad (x \in U(a),\ x \neq a).$$

Whence for the indicated values of x it follows that

$$|f(x)| \leqslant 1 + |A|,$$

where $1 + |A|$ plays the role of M.

The theorem has been proved.

Theorem 2. *If* $\lim\limits_{x\to a} f(x) = A$ *and* $A \neq 0$ *is a finite number, then there is a neighbourhood* $U(a)$ *such that*

$$|f(x)| > |A|/2 \quad (x \in U(a), \quad x \neq a).$$

Moreover, for the indicated values of x *we have*

$$f(x) > A/2, \text{ if } A > 0,$$

and

$$f(x) < A/2, \quad \text{if } A < 0.$$

Proof. It follows from the conditions of the theorem that there is a neighbourhood $U(a)$ such that

$$|A|/2 > |A - f(x)| \geqslant |A| - |f(x)| \quad (x \in U(a), \\ x \neq a),$$

which becomes quite clear if we denote $\varepsilon = |A|/2$. Therefore for the indicated values of x we have $|f(x)| > |A|/2$. The first of the above inequalities can be equivalently replaced by the inequalities

$$A - \frac{|A|}{2} < f(x) < A + \frac{|A|}{2}.$$

Whence, for $A > 0$, it follows

$$\frac{A}{2} = A - \frac{|A|}{2} < f(x),$$

and for $A < 0$ we have

$$f(x) < A + \frac{|A|}{2} = \frac{A}{2},$$

which is what we had to prove.

Theorem 3. *If*

$$\lim_{x \to a} f_1(x) = A_1, \quad \lim_{x \to a} f_2(x) = A_2$$

and if there is a neighbourhood $U(a)$ *such that*

$$f_1(x) \leqslant f_2(x),$$

for $x \in U(a)$, $x \neq a$, *then* $A_1 \leqslant A_2$.

Proof. Let $x_n \to a$, $x_n \neq a$; then for a sufficiently large n_0 we can write the inequality

$$f_1(x_n) \leqslant f_2(x_n) \quad (n > n_0)$$

which, on passing to the limit, results in $A_1 \leqslant A_2$.

Theorem 4. *If*

$$\lim_{x \to a} f_1(x) = A, \quad \lim_{x \to a} f_2(x) = A \tag{1}$$

and if

$$f_1(x) \leqslant \varphi(x) \leqslant f_2(x), \tag{2}$$

for the values of x belonging to a neighbourhood $U(a)$, $x \neq a$, then

$$\lim_{x \to a} \varphi(x) = A. \tag{3}$$

Proof. Let $x_n \to a$, $x_n \neq a$; then for $n > n_0$ we have

$$f_1(x_n) \leqslant \varphi(x_n) \leqslant f_2(x_n)$$

if n_0 is sufficiently large. By virtue of (1), it follows that the limit of $\varphi(x_n)$ exists and is equal to A. Since $\{x_n\}$ is an arbitrary sequence convergent to a, condition (3) does in fact hold.

Theorem 5 (Cauchy's criterion for existence of limit). *For the (finite) limit $\lim\limits_{x \to a} f(x)$ to exist, it is necessary and sufficient that the function $f(x)$ be defined in a neighbourhood of a except, possibly, at the point a itself and that for any $\varepsilon > 0$ there exist a neighbourhood $U(a)$ such that the inequality*

$$| f(x') - f(x'') | < \varepsilon$$

holds for any two points x' and x'' belonging to $U(a)$ and different from a, i.e. x', $x'' \in U(a)$, x', $x'' \neq a$.

Proof. Let $\lim\limits_{x \to a} f(x) = A$, where A is a finite number. Then there is a neighbourhood of a such that $f(x)$ is defined throughout this neighbourhood with the possible exception of the point a itself. Furthermore, given any $\varepsilon > 0$, we can find a neighbourhood $U(a)$ such that if $x \in U(a)$, $x \neq a$, then $| f(x) - A | < \varepsilon/2$. Let x', $x'' \in U(a)$ and x', $x'' \neq a$; then

$$|f(x') - f(x'')| \leqslant |f(x') - A| + |A - f(x'')| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows the necessity of the condition of the theorem.

Let us prove the sufficiency of this condition. Let a function $f(x)$ be defined in a neighbourhood of a except, possibly, at the point a itself and let for any $\varepsilon > 0$ there exist a neighbourhood $U(a)$ such that $| f(x') - f(x'') | < \varepsilon$ for all x', $x'' \in U(a)$, x', $x'' \neq a$. Taking an arbitrary sequence $\{x_n\}$, $x_n \neq a$ $(n = 1, 2, \ldots)$ tending to a, we can assert that, according to Cauchy's criterion for convergence of a sequence, there is a number n_0 such that

$x_n, x_m \in U(a)$ for $n, m > n_0$. Then

$$|f(x_n) - f(x_m)| < \varepsilon \quad (n, m > n_0).$$

And, consequently, the sequence $\{f(x_n)\}$ satisfies Cauchy's criterion and thus has a limit.

We have proved the following property of the function f under consideration: for any sequence of numbers $x_n \neq a$ convergent to a there exists the limit $\lim f(x_n)$. This property implies automatically that all these limits $\lim f(x_n)$ corresponding to all the possible different sequences tending to a are equal to each other. This means that the limit $\lim\limits_{x \to a} f(x)$ exists. Indeed, let $x_n \to a$, $x'_n \to a$; $x_n, x'_n \neq a$ $(n = 1, 2, \ldots)$. Then, as we have shown, there are numbers A and A' such that $f(x_n) \to A$ and $f(x'_n) \to A'$. Let us construct a new sequence: $\{x_1, x'_1, x_2, x'_2, x_3, \ldots\}$. It converges to a. By what has already been proved, the corresponding sequence $\{f(x_1), f(x'_1), f(x_2), f(x'_2), \ldots\}$ should also converge, which is possible only if $A = A'$. Hence, $A = A'$.

The theorem has been proved.

Theorem 6. *Let*

$$\lim_{x \to a} f(x) = A, \quad \lim_{x \to a} \varphi(x) = B,$$

where A and B are finite numbers. Then

$$\lim_{x \to a} [f(x) \pm \varphi(x)] = A \pm B, \quad \lim_{x \to a} [f(x)\varphi(x)] = AB$$

and if $B \neq 0$, then also

$$\lim_{x \to a} \frac{f(x)}{\varphi(x)} = \frac{A}{B}.$$

As an instance, let us prove the second equality. Let $x_n \to a$, $x_n \neq a$ $(n = 1, 2, \ldots)$; then

$$\lim f(x_n) = A, \quad \lim \varphi(x_n) = B.$$

Since the limit of the product of two variables running through some sequences is equal to the product of the corresponding limits, we have

$$\lim [f(x_n)\varphi(x_n)] = \lim f(x_n) \lim \varphi(x_n) = AB.$$

This equality has been proved for any variable $x_n \to a$, $x_n \neq a$, and therefore $\lim\limits_{x \to a} [f(x)\varphi(x)] = AB$.

According to the definition, $\lim\limits_{x \to a} f(x) = \infty$ if the function $f(x)$ is defined in a neighbourhood of a except,

possibly, at the point a itself and if for any positive number M there is a neighbourhood $U(a)$ of the point a such that

$$|f(x)| > M \quad (x \in U(a), \quad x \neq a).$$

A function for which $\lim_{x \to a} f(x) = \infty$ is called *infinitely large as* $x \to a$.

If $\lim_{x \to a} f(x) = \infty$ and if the function $f(x)$ satisfies the inequality $f(x) > 0$ $(f(x) < 0)$ in a neighbourhood of a, then we also write

$$\lim_{x \to a} f(x) = +\infty \quad (\lim_{x \to a} f(x) = -\infty).$$

It is easy to prove the following theorems.

Theorem 7. *If a function $f(x)$ satisfies the inequality*

$$|f(x)| > M > 0$$

in a neighbourhood of a and if a function $\varphi(x)$ is such that

$$\lim_{x \to a} \varphi(x) = 0 \quad (\varphi(x) \neq 0 \textit{ for } x \neq a),$$

then

$$\lim_{x \to a} \frac{f(x)}{\varphi(x)} = \infty.$$

Theorem 8. *If* $\lim_{x \to a} f(x) = A$, $\lim_{x \to a} \varphi(x) = \infty$ (*where A is a finite number*), *then*

$$\lim_{x \to a} \frac{f(x)}{\varphi(x)} = 0.$$

Corollary. *If* $\varphi(x) \to 0$ $(x \to a,\ \varphi(x) \neq 0)$, *then*

$$\lim_{x \to a} \frac{1}{\varphi(x)} = \infty,$$

and if $\varphi(x) \to \infty$ $(x \to a,\ \varphi(x) \neq 0)$, *then*

$$\lim_{x \to a} \frac{1}{\varphi(x)} = 0.$$

We can also define the right-hand and left-hand limits of a function f at a (finite) point a.

A number A is called the right-hand (left-hand) limit of a function f at a point a if it is defined on a certain half-interval $(a, b]$ ($[b, a)$) and if there exists

$$\lim_{\substack{x_n \to a \\ x_n > a}} f(x_n) = A \quad \left(\text{respectively} \lim_{\substack{x_n \to a \\ x_n < a}} f(x_n) = A\right)$$

for any indicated sequence $\{x_n\}$.

The right-hand and left-hand limits of function f at a point a are usually denoted as follows:

$$f(a+0) = \lim_{\substack{x \to a \\ x > a}} f(x), \tag{4}$$

$$f(a-0) = \lim_{\substack{x \to a \\ x < a}} f(x). \tag{5}$$

If f is defined on an open interval (a, b), then at the point a only the number $f(a+0)$ makes sense, and at the point b only the number $f(b-0)$ does.

Remark. The equalities

$$f(a+0) = f(a-0) = A \tag{6}$$

are equivalent to the existence of the limit

$$\lim_{x \to a} f(x) = A. \tag{7}$$

Indeed, (6) can be expressed in the following way: $\forall \varepsilon > 0, \quad \exists \delta > 0: \ | f(x) - A | < \varepsilon, \quad \forall x: \ 0 < | x - a | < \delta, \ x > a; \quad | f(x) - A | < \varepsilon, \quad \forall x: \ 0 < | x - a | < \delta, \ x < a$. But this can be expressed by a briefer notation: $\forall \varepsilon > 0, \quad \exists \delta > 0: \ | f(x) - A | < \varepsilon, \ \forall x: \ 0 < | x - a | < \delta$, which is equivalent to (7).

Sec. 3.3. Continuity of a Function

Figure 14 represents the graph of a function $y = f(x)$ $(a \leqslant x \leqslant b)$. It is natural to call this graph continuous because it can be drawn by a continuous motion of the pencil without leaving the paper. Let us take an arbitrary point (number) $x \in [a, b]$. Another point $x' \in [a, b]$ lying close to the former can be written in the form $x' = x + \Delta x$, where Δx is a positive or

negative number called the *increment of* x. The difference

$$\Delta f = \Delta y = f(x + \Delta x) - f(x)$$

is called the *increment of the function* f, at the point x, corresponding to the increment Δx of the independent variable x. Here is meant Δx such that $x + \Delta x \in [a, b]$.

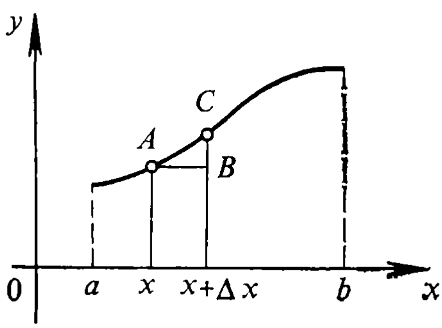

Fig. 14

In Figure 14 the quantity Δy is equal to the length of the line segment BC.

Let us make Δx continuously tend to zero; then, for the function under consideration, Δy will also tend to zero:

$$\Delta y \to 0 \quad (\Delta x \to 0). \quad (1)$$

Now let us consider the graph shown in Fig. 15. It consists of two continuous pieces PA and QR. But these pieces are not connected to form a continuous curve and therefore it appears natural to call this graph discontinuous. In order for this graph to represent a one-valued function $y = F(x)$ at the point x_0, let us agree that $F(x_0)$ is equal to the length of the line segment joining A and x_0; to indicate this convention, we have shown the point A in the figure as a small circle, while the point Q is shown as placed at the tip of an arrow which symbolizes that the latter point does not belong to the graph. If the point Q belonged to the graph, then the function F would be two-valued at the point x_0.

Let us now give x_0 an increment Δx_0 and determine the corresponding increment of the function:

$$\Delta F = F(x_0 + \Delta x) - F(x_0).$$

If Δx_0 is made to tend to zero continuously, then we may no longer assert that ΔF also tends to zero. This is the case for the negative Δx_0 tending to zero, while for the positive Δx_0 this is not so; as is seen in the figure, if Δx_0 tends to zero retaining the positive sign, then the corresponding increment ΔF tends to a positive number equal to the length of the line segment AQ.

These considerations lead us to the following definition: *a function f defined on a closed interval $[a, b]$ is said to be continuous at a point x belonging to that interval if the increment of the function at the point x, corresponding to the increment Δx of the argument, tends to zero as Δx is made to tend to zero in an arbitrary way.* This property of continuity of f at x is expressed by relation (1) and can also be written as

$$\lim_{\Delta x \to 0} \Delta y = 0. \qquad (2)$$

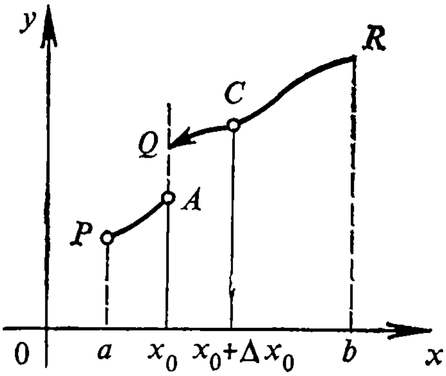

Fig. 15

Relation (2) reads: the limit of Δy is equal to zero as Δx tends to zero according to any law. By the way, the words "according to any law" are usually omitted, but are, of course, tacitly implied.

If a function f defined on $[a, b]$ is not continuous at a point $x \in [a, b]$, i.e. if property (2) does not hold at x for at least one way of making Δx tend to zero, then the function f is said to be *discontinuous* at the point x.

The function represented in Fig. 14 is continuous at any point $x \in [a, b]$, while the function whose graph is shown in Fig. 15 is obviously, continuous at any point $x \in [a, b]$, except at the point x_0, since for the latter point relation (2) does not hold when Δx tends to zero retaining the positive sign.

A function continuous at every point of a closed (or open) interval is said to be *continuous on that interval.*

A continuous function expresses mathematically a property which is frequently encountered in practice: to a small increment of an independent variable there also corresponds a small increment of another variable (function) dependent on the former. Perfect examples of continuous functions are various laws of motion of physical bodies $s = f(t)$ expressing the dependence of the paths s travelled by the bodies in time t. Time and space are continuous, and a given law of motion $s = f(t)$

establishes a certain continuous dependence between them whose characteristic feature is that to a small increment of time there corresponds a small increment of the path.

We come to the abstract notion of continuity observing the so-called continuous media, i.e. rigid bodies, liquids and gases, such as metals, water, and air. Every real physical medium is in fact an accumulation of a large number of discrete moving particles. But these particles and the distances between them are so small, compared with the volumes occupied by the media we deal with in considering macroscopic physical phenomena, that many of such phenomena can be approximated sufficiently well if we consider the masses of the media in question as being distributed in space continuously without any "gaps" between the particles. Such an approach forms the basis of many physical disciplines, such as hydrodynamics, aerodynamics, and the theory of elasticity. The mathematical concept of continuity naturally plays an important role in these and many other fields of science.

Continuous functions constitute the basic group of functions dealt with in mathematical analysis.

Examples of continuous functions are the elementary functions discussed below in Sec. 3.8. They are all continuous on the intervals of variation of x where they are defined.

Discontinuous functions which are also studied in mathematics describe abrupt processes observed in reality. For instance, the velocities of the colliding bodies change in an abrupt manner. Many processes of qualitative transition are accompanied with jump phenomena. For example, the dependence $Q = f(t)$ between the temperature t of 1 g of water (ice) and the quantity Q of calories of heat contained in it corresponding to the variation of t from $-10°$ to $+30°$ is described, if we agree, conventionally, that $Q = 0$ for $t = -10°$, by the formulas:

$$Q(t) = \begin{cases} 0.5t + 5, & -10 \leqslant t < 0, \\ t + 85, & 0 < t \leqslant 30. \end{cases}$$

Here we have assumed that the specific heat of ice is equal to 0.5. The function $Q = f(t)$ is not defined in a unique manner (i.e. it is multiple-valued) for $t = 0$. For the sake of convenience, we can agree that at the point $t = 0$ it takes on a definite value, say, $f(0) = 45$. The considered function $Q = f(t)$ is, obviously, discontinuous at the point $t = 0$; its graph is shown in Fig. 16.

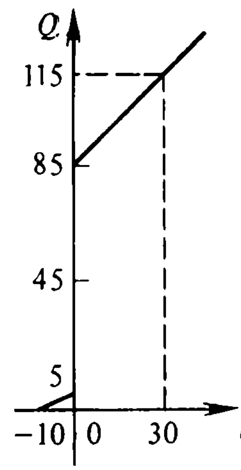

Fig. 16

Let us define the continuity of a function f at a point.

A function $f(x)$ is said to be continuous at a point x_0 if it is defined in some neighbourhood of this point (including the point x_0 itself) and if its increment at this point, corresponding to the increment of the argument Δx, tends to zero as $\Delta x \to 0$:

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} [f(x_0 + \Delta x) - f(x_0)] = 0. \tag{3}$$

Setting $x = x_0 + \Delta x$, we obtain the following equivalent definition of the continuity of f at x_0: *a function f is said to be continuous at a point x_0 if it is defined at some neighbourhood of that point, including the point x_0 itself, and if*

$$\lim_{x \to x_0} f(x) = f(x_0); \tag{4}$$

or in terms of ε and δ: if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon, \quad \forall x: |x - x_0| < \delta$$

Equality (4) can also be written in the following way:

$$\lim_{x \to x_0} f(x) = f\left(\lim_{x \to x_0} x\right). \tag{4'}$$

It shows that *under the symbol of a continuous function it is possible to pass to the limit.*

Example 1. The constant $y = C$ is a function continuous at any point x. Indeed, to the point x there corresponds the value of the function $y = C$, to the point

$x + \Delta x$ there corresponds the same value: $y(x + \Delta x) = C$. Therefore $\Delta y = y(x + \Delta x) - y(x) = C - C = 0$ and

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} 0 = 0.$$

Example 2. The function $y = x$ is continuous for any value of x since $\Delta y = \Delta x$ and, consequently, $\Delta y \to 0$ as $\Delta x \to 0$.

Example 3. The function $y = \sin x$ is continuous for any x.

Indeed,

$$|\Delta y| = |\sin(x + \Delta x) - \sin x| = \left|2 \sin \frac{\Delta x}{2} \cos\left(x + \frac{\Delta x}{2}\right)\right| \leqslant 2\,|\sin(\Delta x/2)|. \qquad (5)$$

But for any α the following inequality is valid:

$$|\sin \alpha| < |\alpha|. \qquad (6)$$

If $0 < \alpha < \pi/2$, then this follows from Fig. 17, where a circle of radius 1 is represented (the chord of length $2 \sin \alpha$ subtending an arc of length 2α is shorter than this arc). For $\alpha = 0$ inequality (6) turns into an equality. And if $0 < |\alpha| < \pi/2$, then $|\sin \alpha| = \sin|\alpha| \leqslant |\alpha|$. Finally, if $|\alpha| > \pi/2$, then $|\sin \alpha| \leqslant 1 < \pi/2 \leqslant |\alpha|$. By virtue of (6), it follows from (5):

$$|\Delta y| \leqslant 2\left|\sin \frac{\Delta x}{2}\right| \leqslant 2\,\frac{|\Delta x|}{2} = |\Delta x|,$$

i.e.

$$|\Delta y| \leqslant |\Delta x|.$$

But then, obviously,

$$\lim_{\Delta x \to 0} \Delta y = 0.$$

We can also say that for any $\varepsilon > 0$ we can find $\delta > 0$, viz. $\delta = \varepsilon$ such that

$$|\Delta y| < \varepsilon, \qquad \forall \Delta x: \quad |\Delta x| < \delta = \varepsilon.$$

Let us note an important theorem.

Theorem 1. *If functions f and φ are continuous at a point $x = a$, then their sum, difference, product, and quotient (for $\varphi(a) \neq 0$) are also continuous at this point.*

This theorem follows directly from Theorem 6 proved in the preceding section, taking into consideration that in the present case

$$f(a) = \lim_{x \to a} f(x), \quad \varphi(a) = \lim_{x \to a} \varphi(x).$$

An important theorem on the continuity of a function of a function (of a composite function) is also valid.

Theorem 2. *Let there be given a function $f(u)$ continuous*

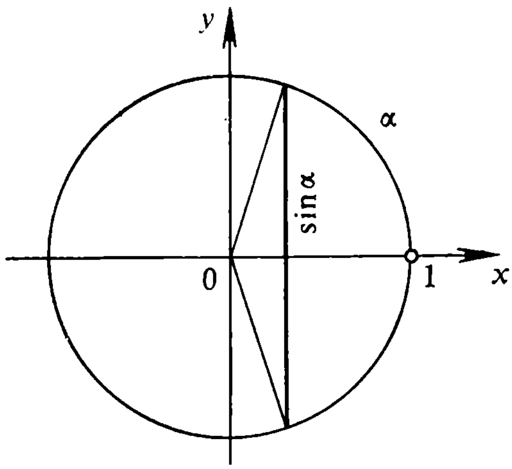

Fig. 17

at a point $u = A$ and another function $u = \varphi(x)$ continuous at a point $x = a$, and let $\varphi(a) = A$. Then the composite function $F(x) = f[\varphi(x)]$ is continuous at the point $x = a$.

Proof. Note, that according to the definition of the continuity of a function f at a point A, it follows that it is defined in some neighbourhood of this point. Therefore

$$\lim_{x \to a} F(x) = \lim_{x \to a} f[\varphi(x)] = \lim_{u \to A} f[u]$$
$$= f(A) = f[\varphi(a)] = F(a).$$

Here we have introduced the substitution $u = \varphi(x)$ and have taken into account the continuity of φ at the point $x = a$: $\varphi(x) \underset{x \to a}{\to} \varphi(a) = A$.

Example 4. The function

$$P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n,$$

where a_k are constant coefficients, is called a *polynomial* of degree n. It is continuous for any x. In fact, to obtain $P(x)$, we have, proceeding from the constant numbers $a_0, \ldots, a_n$ and function x, to accomplish a finite number of arithmetic operations: addition, subtraction, and multiplication. But a constant is a continuous function (see Example 1), and the function $y = x$ is also continuous (see Example 2), therefore the continuity of $P(x)$ follows from Theorem 1.

Example 5. The function $y = \cos x$ is continuous. It is a composition of two continuous functions: $y = \sin u$, $u = \frac{\pi}{2} - x$.

Example 6. The function

$$y = \tan x = \frac{\sin x}{\cos x}, \quad x \neq \frac{\pi}{2} + k\pi \quad (k = 0, \pm 1, \pm 2, \ldots),$$

is continuous for the indicated x's, since (see Theorem 1) it equals to a quotient obtained as a result of division of two continuous functions, the divisor being not equal to zero (for the given x's).

Example 7. The function

$$y = \sin^3 x^5$$

is continuous for any x, since it is a composition of the following continuous functions: $y = u^3$, $u = \sin v$, $v = x^5$ (see Theorem 2).

Example 8. The function $y = |x|$ is continuous for any x, since

$$|\Delta y| = ||x + \Delta x| - |x|| \leqslant |x + \Delta x - x| = |\Delta x| \to 0$$

as $\Delta x \to 0$.

Example 9. If a function $f(x)$ is continuous at a point x_0, then the function $|f(x)|$ is also continuous at this point.

This follows from Theorem 2 and Example 8, since the latter function is a composition of two continuous functions: $y = |u|$, $u = f(x)$.

Let us mention two more theorems which directly follow from Theorems 1 and 2 on the limit of a function proved in the preceding section.

Theorem 3. *If a function f is continuous at a point a, then there exists a neighbourhood $U(a)$ of this point in which f is bounded.*

Theorem 4. *If a function f is continuous at a point a and $f(a) \neq 0$, then there exists a neighbourhood $U(a)$ of the point a in which*

$$|f(x)| > |f(a)|/2.$$

Moreover, if $f(a) > 0$, then

$$f(a)/2 < f(x) \quad (x \in U(a)),$$

and if $f(a) < 0$, then

$$f(x) < f(a)/2 \quad (x \in U(a)).$$

Sec. 3.4. Discontinuities of the First and Second Kind

By definition, *a function f is continuous from the right (left) at a point $x = a$ if*

$$f(a) = f(a+0) \quad \text{(respectively,} \quad f(a) = f(a-0))$$

(see the closing part of Sec. 3.2).

The continuity of f at a point a can also be defined in the following way: *a function f is continuous at a point $x = a$ if it is defined in a certain neighbourhood of this point (including the point $x = a$ itself) and there exist limits $f(a+0)$ and $f(a-0)$ such that*

$$f(a) = f(a+0) = f(a-0). \tag{1}$$

If a function f is such that there exist the limits $f(a+0)$, $f(a-0)$ for it, but equalities (1) are not fulfilled, then, obviously, it is *discontinuous* (not continuous) at the point a. In this case the function f is said to have a *discontinuity of the first kind* at the point a.

In Figures 18-23 we see six graphs representing functions having a discontinuity of the first kind at a point a. Here the letter A denotes the point $A = (a, f(a))$ in the xy-plane. An arrow placed at an end point of a piece of a curve symbolizes that this end point is deleted from the graph.

In Figures 18-21 we see the cases when all the three numbers $f(a)$, $f(a+0)$, and $f(a-0)$ make sense for

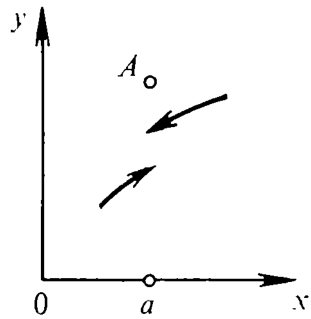

Fig. 18

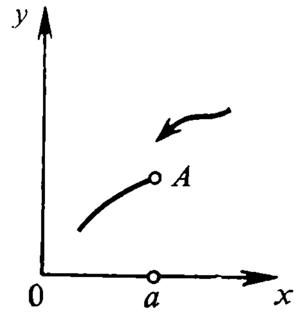

Fig. 19

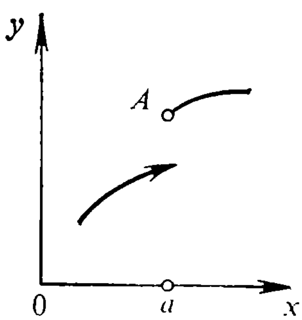

Fig. 20

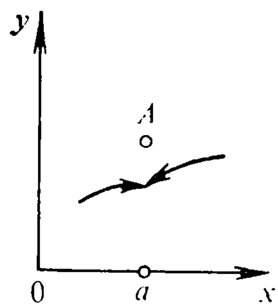

Fig. 21

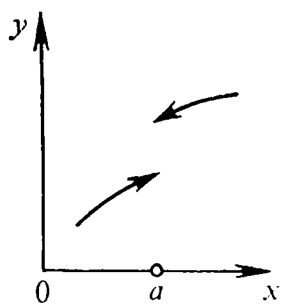

Fig. 22

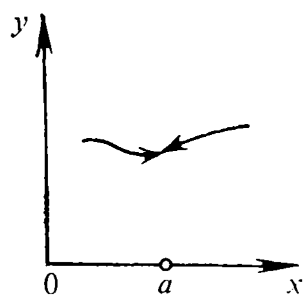

Fig. 23

the depicted graphs of certain functions f. In Figure 18 the three numbers $f(a)$, $f(a+0)$, and $f(a-0)$ are pairwise different; the corresponding function is discontinuous not only at a, but also from the right and from the left at a. In Figure 19 the function is continuous from the left at a, but is discontinuous from the right at this point. In Figure 21 we see that $f(a+0) =$

$= f(a-0) \neq f(a)$. In such a case the function f is said to have a *removable discontinuity* at the point a. As a matter of fact, we can modify it at the point a by setting $f(a) = f(a+0) = f(a-0)$, and the function will become continuous at this point. In Figure 22 the function is not defined at a. In Figure 23 the function is not defined at a either, but $f(a+0) = f(a-0)$, therefore if we extend f to a by setting $f(a) = f(a+0) = f(a-0)$, then the function f will become continuous at the point a.

In Figures 22 and 23 the function f is defined in a neighbourhood of the point except for the point a itself. In such cases we often say that f is discontinuous at a, although the idea of continuity and discontinuity of f at a is an idea of comparison of $f(a)$ with $f(x)$ for x's close to a.

If a function f has no right-hand or left-hand limit at a point a, or neither right-hand, nor left-hand limit, or these limits are infinite, then the function is said to have a *discontinuity of the second kind at this point.*

Example 1. The function

$$f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

has no right-hand and left-hand limits at the point $x = 0$ (see Example 3 given in Sec. 3.2). Consequently, it has a discontinuity of the second kind at this point.

Example 2. The function

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases}$$

is, obviously, continuous for $x \neq 0$ and has a discontinuity of the first kind at the point $x = 0$. Hence, $\operatorname{sgn}(0+0) = 1$ and $\operatorname{sgn}(0-0) = -1$.

Example 3. The function $[x]$ (integral part of x) for $x \geqslant 0$ has a graph shown in Fig. 24. It is continuous for nonintegral x's, and if x is integral, then $[x+0] =$

$= x = [x]$ and $[x - 0] = x - 1$, and, consequently, a discontinuity of the first kind occurs.

Example 4. The function

$$y = \begin{cases} 1/x, & x \neq 0, \\ 2, & x = 0, \end{cases}$$

is continuous for $x \neq 0$. The right-hand and left-hand limits at the point $x = 0$ are equal to infinity, therefore the function has a discontinuity of the second kind at this point. In this case the function is said to have an *infinite discontinuity at this point.*

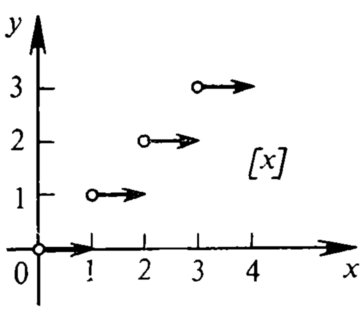

Fig. 24

Theorem 1. *If a function f does not decrease on a closed interval $[a, b]$, then thered exist limits $f(a + 0) \geqslant f(a)$* and $f(b - 0) \leqslant f(b)$.

Proof. It follows from the given condition that

$$f(x) \leqslant f(b), \qquad \forall x \in [a, b),$$

i.e. f is bounded above by the number $f(b)$ on the half-interval $[a, b)$. But then there exists the supremum of f on this half-interval:

$$\sup_{x \in [a, b)} f(x) = M \leqslant f(b).$$

By virtue of the property of the supremum, for any $\varepsilon > 0$ there is $x_0 \in [a, b)$ such that

$$M - \varepsilon < f(x_0) \leqslant M, \tag{2}$$

and by virtue of the fact that f does not decrease, the following is valid:

$$f(x_0) \leqslant f(x), \quad \forall x\colon x_0 < x < b. \tag{3}$$

From (2) and (3) it follows that

$$M - \varepsilon < f(x) \leqslant M, \quad \forall x\colon x_0 < x < b,$$

and. thus, we have proved that there exists the left-hand limit of f at the point b:

$$\lim_{\substack{x\to b\\ x<b}} f(x)=f(b-0)=M\leqslant f(b).$$

Analogously, considering the inequality $f(a)\leqslant f(x)$ for $x\in(a, b]$ we prove the existence of

$$f(a+0)=\inf_{x\in(a,\ b]} f(x)\geqslant f(a).$$

Corollary. *If a function f is nondecreasing on a closed interval $[a, b]$, then at any point $x\in[a, b)$ there exists the right-hand limit $f(x+0)\geqslant f(x)$ and at any point $x\in(a, b]$ there exists the left-hand limit $f(x-0)\leqslant f(x)$.*

Indeed, for the point $x=a, b$ this statement is proved in Theorem 1. Let $x\in(a, b)$. On $[a, x]$ and $[x, b]$ the function f is nondecreasing and, therefore, by Theorem 1, there exist the limits $f(x-0)$, $f(x+0)$ and $f(x-0)\leqslant f(x)\leqslant f(x+0)$.

In the present case, obviously, for the function f to be continuous at the point x, it is necessary and sufficient that $f(x-0)=f(x+0)$.

If $f(x-0)<f(x+0)$, then the function f has a discontinuity of the first kind at the point x.

Theorem 2. *The set of points of discontinuity of a function f monotone on a closed interval $[a, b]$ is at most countable.*

Proof. Let the function f have more than one point of discontinuity and let x' and x'' $(x'<x'')$ be two of them. Since

$$f(x'+0)=\inf_{y\in(x',\ x'')} f(y),\quad f(x''-0)=\sup_{y\in(x',\ x'')} f(y)$$

then

$$f(x'+0)\leqslant f(x''-0)$$

and the intervals $(f(x'-0), f(x'+0))$, $(f(x''-0), f(x''+0))$ of the y-axis do not intersect.

To each point x' of discontinuity of the function f there corresponds the interval $(f(x'-0), f(x'+0))$. Let us choose a rational point $\alpha_{x'}$ inside this interval. After the aforesaid, it is clear that to different points of discontinuity x' there correspond different points $\alpha_{x'}$. But the set of all rational numbers is countable, therefore the set of all points $\alpha_{x'}$, the same as the set of all points x' (of discontinuity of the function f), is at most countable. The theorem has been proved.

Sec. 3.5. Functions Continuous on a Closed Interval

A function f is said to be *continuous on a closed interval* $[a, b]$ if it is continuous at all the points of the open interval (a, b), is continuous from the right at the point a and continuous from the left at the point b.

Functions continuous on a closed interval possess a number of remarkable properties which will be set forth in this section.

First we shall formulate the theorems expressing these

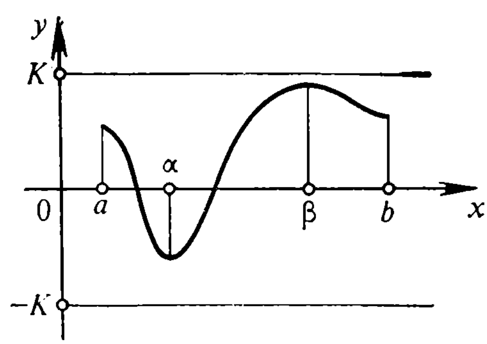

Fig. 25

properties and explain them with the aid of graphs and examples, and then prove them formally.

Theorem 1. *If a function f is continuous on a closed interval $[a, b]$, then it is bounded on this interval, i.e. there exists a constant $K > 0$ such that the following inequality is fulfilled:*

$$|f(x)| \leqslant K, \quad \forall x \in [a, b].$$

Represented in Fig. 25 is the graph Γ of a continuous function f on a closed interval $[a, b]$. Obviously, there exists a number $K > 0$ such that Γ is located below the straight line $y = K$, but above the line $y = -K$. This is just what is meant by Theorem 1.

Note that if a function is continuous on an open interval (a, b) or on a half-interval $[a, b)$ or $(a, b]$, then it is not necessarily bounded on such an interval. For instance, the function $1/x$ is continuous on the half-interval $(0, 1]$, but is not bounded on it.

If we supplement the definition of this function by setting $f(0) = 0$, then it will be finite at any point of the closed interval $[0, 1]$, but not bounded on this interval.

Theorem 2 (Weierstrass' Theorem). *If a function f is continuous on a closed interval $[a, b]$, then there exist its minimum and maximum on $[a, b]$, i.e. there exist points $\alpha, \beta \in [a, b]$ such that $f(\alpha) \leqslant f(x) \leqslant f(\beta)$ for all $x \in [a, b]$. In other words,*

$$\min_{x\in[a, b]} f(x) = f(\alpha), \quad \max_{x\in[a, b]} f(x) = f(\beta).$$

The continuous function $y = f(x)$ graphed in Fig. 25 attains its minimum on $[a, b]$ at the point $x = \alpha$ and maximum at the point $x = \beta$. In this case both points α and β belong to the interval (a, b) $(\alpha, \beta \in (a, b))$. The continuous function $y = f(x)$ represented in Fig. 26 reaches its minimum on $[a, b]$ at its left-hand end point and maximum at a certain interior point β of this interval.

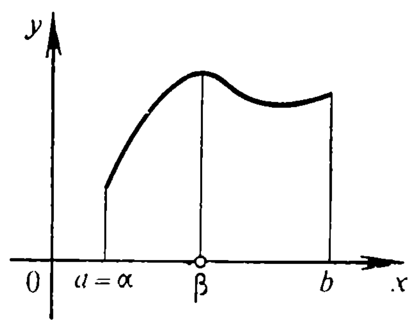

Fig. 26

Remark 1. According to Theorem 1, a function continuous on a closed interval $[a, b]$ is bounded on it. Consequently, there exist finite infimum and supremum of f on this interval:

$$\inf_{x\in[a, b]} f(x) \leqslant \sup_{x\in[a, b]} f(x).$$

Theorem 2 states that they are attained on $[a, b]$, i.e. here inf and sup can be replaced by min (minimum) and max (maximum), respectively.

Remark 2. The function $y = x$ is continuous on the open interval $(0, 1)$ and is bounded on it; its $\sup\limits_{x\in(0,1)} x = 1$ is not attained, i.e. there is no such $x_0 \in (0, 1)$ for which this function is equal to 1. Thus, in Theorem 2 the condition of the continuity of f on a *closed interval* (containing both of its end points a and b) is essential.

It is obvious that $\sup\limits_{x \geqslant 0} \arctan x = \pi/2$. But there is no such x on the ray $x \geqslant 0$ for which the function $\arctan x$ takes on the value $\pi/2$, and it does not attain maximum on $x \geqslant 0$. In this case the conditions of the theorem are not fulfilled: the domain of definition of the continuous function $\arctan x$ is unbounded.

If a function f is discontinuous on $[a, b]$, then it not necessarily attains its supremum. This is exemplified by the function

$$f(x) = \begin{cases} x, & 0 \leqslant x < 1/2, \\ 0, & 1/2 \leqslant x \leqslant 1. \end{cases}$$

Theorem 3. *If a function f is continuous on a closed interval $[a, b]$ and the numbers $f(a)$ and $f(b)$ are different*

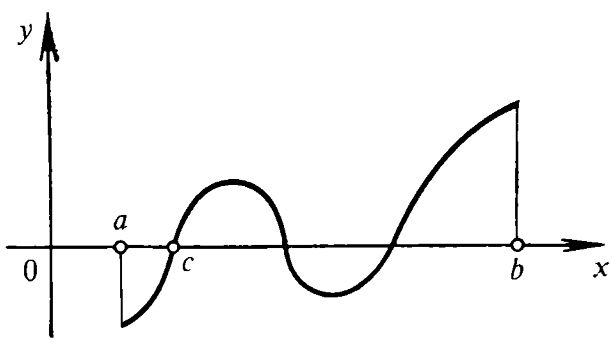

Fig. 27

from zero and have opposite signs, then there is at least one point c on the open interval (a, b) such that $f(c) = 0$.

The function whose graph Γ is depicted in Fig. 27 satisfies the conditions of Theorem 3. It is continuous on $[a, b]$ and $f(a) < 0$, $f(b) > 0$. Geometrically, it is obvious that the graph Γ must intersect the x-axis at least at one point $c \in (a, b)$. This is just what is stated by Theorem 3.

Corollary. *If a function f is continuous on $[a, b]$, $f(a) = A$, $f(b) = B$ $(A \neq B)$, and C is an arbitrary number lying between the numbers A and B, then there is at least one point c belonging to the open interval (a, b) such that $f(c) = C$.*

This corollary can also be stated as follows: *a function continuous on a closed interval* $[a, b]$ *assumes all the intermediate values lying between its end-point values.*

It can also be formulated in the following way: *a function continuous on a closed interval* $[a, b]$ *assumes all the intermediate values lying between its least and greatest values on* $[a, b]$ (which exist by virtue of Theorem 2).

Proof. Let us define a new function $F(x) = f(x) - C$, where C is a constant number lying between A and B. Since f is a continuous function on $[a, b]$, so is the function F. The function F, obviously, attains values of opposite signs at the end points of the interval $[a, b]$. Therefore, by Theorem 3, there is a point c lying inside (a, b) such that $F(c) = 0$ or $f(c) - C = 0$, that is, $f(c) = C$. This is what we wished to prove.

Example 1. The equation $x - \cos x = 0$ possesses a root lying within the interval $(0, \pi)$.

Indeed, the function $f(x) = x - \cos x$ is continuous on the closed interval $[0, \pi]$ and assumes the $f(0) = -1$, and $f(\pi) = \pi + 1$ values having opposite signs at its end points.

Below, we prove Theorems 1-3 in a formal manner.

The Proof of Theorem 1. Suppose that f is unbounded on $[a, b]$. Then for every natural number n we can find a point $x_n \in [a, b]$ such that

$$|f(x_n)| > n \quad (n = 1, 2, \ldots). \tag{1}$$

The sequence $\{x_n\}$ is bounded (a and b are finite numbers!) and therefore it contains a subsequence $\{x_{n_k}\}$, convergent to a point $\alpha \in [a, b]$ (see the corollary of Theorem 4 proved in Sec. 2.1). But the function f is continuous at the point α (if $\alpha = a$ $(\alpha = b)$, then f is continuous from the right (left) at this point), and therefore

$$\lim_{k \to \infty} f(x_{n_k}) = f(\alpha). \tag{2}$$

Property (2) contradicts property (1). Therefore f must be bounded on $[a, b]$.

The Proof of Theorem 2. By the preceding theorem, a continuous function on a closed interval $[a, b]$ is bounded, and, consequently, this function is bounded

above by some number K

$$f(x) \leqslant K \quad (x \in [a. \ b]).$$

But then there exists the supremum of f on $[a, b]$:

$$\sup_{x \in [a, b]} f(x) = M. \tag{3}$$

The number M possesses the following property: for any natural number n there is a point x_n on $[a, b]$ such that

$$M - \frac{1}{n} < f(x_n) \leqslant M \quad (n = 1, 2, \ldots).$$

The sequence $\{x_n\}$ belongs to $[a, b]$ and is therefore bounded. Hence it contains a subsequence $\{x_{n_k}\}$ convergent to a number β which is sure to belong to $[a, b]$. But the function f is continuous at the point β and therefore

$$\lim_{k \to \infty} f(x_{n_k}) = f(\beta).$$

On the other hand, $M - \frac{1}{n_k} < f(x_{n_k}) \leqslant M$ $(k = 1, 2, \ldots)$ and

$$\lim_{k \to \infty} f(x_{n_k}) = M.$$

But since $f(x_{n_k})$ can tend only to one limit, we have $M = f(\beta)$. The value equal to supremum (3) is thus attained by the function at the point β; in such a case, the *function f is said to attain at the point β its maximum* (*or maximum value*) *on the closed interval* $[a, b]$. And so, we have proved that there exists a point $\beta \in [a, b]$, for which

$$\max_{x \in [a, b]} f(x) = f(\beta).$$

The remaining part of the theorem concerning the minimum of the function is proved in a similar way. It can also be reduced to the first part of the theorem, since

$$\min_{x \in [a, b]} f(x) = - \max_{x \in [a, b]} \{-f(x)\}.$$

The Proof of Theorem 3. Let us denote the closed interval $[a, b]$ by σ_0. On dividing it into two equal parts, we test the value of the function f at the midpoint. If the function is equal to zero at the midpoint of σ_0, then the proof is completed. Let us assume that it is different from zero at that point, then one of the two halves of the interval σ_0 is such that the function takes on nonzero values of opposite signs at its end points. Now we denote that very half by σ_1 and again divide it into two equal parts. It may happen that the function is equal to zero at the midpoint of σ_1 and then the theorem is proved. If otherwise, we denote by σ_2 the half of σ_1 at whose end points the values of f are different from zero and have different signs, and so on. Proceeding in this way, we conclude, by induction, that we either encounter a point $c \in (a, b)$ for which $f(c) = 0$, and then the proof of the theorem is completed, or arrive at the sequence of nested intervals $\sigma_0 \supset \sigma_1 \supset \sigma_2 \supset \ldots$, on each of which the function f assumes values of different signs. Then there exists a (single) point c belonging to all σ_n, and, consequently, belonging to $[a, b]$. Obviously, $f(c) = 0$, since, if otherwise, for instance, if $f(c) > 0$, then we could find a neighbourhood $U(c)$ of the point c such that for all x's from $[a, b]$ belonging to $U(c)$ the function $f(c)$ would be positive. But this is impossible, since for a sufficiently large n the interval $\sigma_n \subset U(c)$, and the function f cannot retain its sign on σ_n. The theorem has been proved.

Sec. 3.6. Inverse of a Continuous Function

Let us consider a continuous function $y = f(x)$ strictly increasing on a closed interval $[a, b]$ (Fig. 28). Let

$$f(a) = \alpha, \quad f(b) = \beta.$$

The graph of this function is a continuous curve. As is seen from the graph, if x increases continuously from a to b, then y also increases continuously from α to β, running once through the values contained in the closed interval $[\alpha, \beta]$. But then to every value $y \in [\alpha, \beta]$ there corresponds a unique value $x \in [a, b]$ such that $y =$

$= f(x)$. This defines on the closed interval $[\alpha, \beta]$ the function

$$x = g(y), \quad y \in [\alpha, \beta],$$

which is called the *inverse of the function* $y = f(x)$.

Obviously, the function $x = g(y)$ is strictly increasing on $[\alpha, \beta]$ and maps this interval onto $[a, b]$. The following identities are fulfilled:

$$f[g(y)] = y, \quad \forall\, y \in [\alpha, \beta],$$
$$g[f(x)] = x, \quad \forall\, x \in [a, b].$$

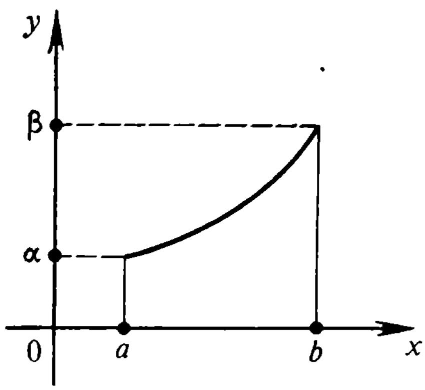

Fig. 28

The graph of the function $x = g(y)$ can be obtained by rotating the plane under consideration about the bisector of the first quadrant of the system x, y through an angle of 180°. Since, as a result of the rotation, the graph remains continuous, this shows that the function $x = g(y)$ is continuous on $[\alpha, \beta]$.

Thus, reasoning geometrically, we have ascertained the validity of the following theorem.

Theorem 1. *Let f be a continuous strictly increasing function on a closed interval $[a, b]$ and let $\alpha = f(a)$, $\beta = f(b)$.*

Then: (1) *the image of the closed interval $[a, b]$ is the closed interval $[\alpha, \beta]$,* (2) *there exists a function $x = g(y)$, the inverse of f, which is one-valued, strictly increasing, and continuous on $[\alpha, \beta]$.*

The formal proof of Theorem 1 will be based on the following lemma.

Lemma 1. *Let a strictly increasing function $y = f(x)$ map the closed interval $[a, b]$ onto the closed interval $[\alpha, \beta]$,* i.e. $f([a, b]) = [\alpha, \beta]$. *Then f is continuous on $[a, b]$.*

Proof. Let us be given an arbitrary point x_0 belonging for the time being to the open interval (a, b) $(a < x_0 < b)$. By virtue of the fact that f is strictly increas-

ing, the corresponding point $y_0 = f(x_0)$ will belong to the interval (α, β) $(\alpha < y_0 < \beta)$.

Let us take $\varepsilon > 0$ so small that $\alpha < y_0 - \varepsilon < y_0 < y_0 + \varepsilon < \beta$. By the hypothesis, there exist points $x_1, x_2 \in (a, b)$ $(x_1 < x_0 < x_2)$ such that $y_0 - \varepsilon = f(x_1)$, $y_0 + \varepsilon = f(x_2)$.

The interval (x_1, x_2) can be regarded as a neighbourhood of the point x_0 $(x_0 \in (x_1, x_2))$.

Since the function f increases, for $x \in (x_1, x_2)$ we shall have $y_0 - \varepsilon < f(x) < y_0 + \varepsilon$ or $|f(x) - y_0| < \varepsilon$, i.e.

$$|f(x) - f(x_0)| < \varepsilon,$$

and we have proved the continuity of the function f at the point x_0.

If $x_0 = a$ or $x_0 = b$, then we prove the one-sided continuity of the function f in a similar way.

The Proof of Theorem 1. Let $Y = f([a, b])$ be the image of $[a, b]$ via f. Since $\alpha = f(a)$, $\beta = f(b)$ and f increases, we have $\alpha \leqslant f(x) \leqslant \beta$, $\forall x \in [a, b]$, whence it follows that

$$Y \subset [\alpha, \beta]. \tag{1}$$

On the other hand, if y is an arbitrary point of the closed interval $[\alpha, \beta]$, then, by Theorem 3 proved in the preceding section, it belongs to Y, i.e.

$$[\alpha, \beta] \subset Y. \tag{2}$$

(1) and (2) prove statement (1):

$$Y = [\alpha, \beta].$$

Statement (2) now follows from Lemma 1. Indeed, since f is strictly increasing, then on $Y = [\alpha, \beta]$ there exists an inverse strictly increasing function $g(y)$ and it maps the closed interval $[\alpha, \beta]$ onto the closed interval $[a, b]$. But then, by Lemma 1, the function $g(y)$ is continuous.

The theorem has been proved.

Introducing insignificant changes into the above arguments, we can prove the following analogy of Theorem 1.

Theorem 1′. *Let f be a continuous function strictly increasing on (a, b) (or on $[a, b)$, or on $(a, b]$) and let*

$$\alpha = \inf_{x \in (a, b)} f(x), \quad \beta = \sup_{x \in (a, b)} f(x).$$

Then the image of (a, b) (of $[a, b)$, $(a, b]$, respectively) is the open interval (α, β) ($[\alpha, \beta)$, $(\alpha, \beta]$, respectively) and the function $x = g(y)$ (the inverse of f) is one-valued, strictly increasing, and continuous on (α, β) ($[\alpha, \beta)$, $(\alpha, \beta]$).

Remark. A strictly decreasing, continuous on $[a, b]$ (on (a, b)) function $f(x)$ has an inverse which is a strictly decreasing continuous function on $[\beta, \alpha]$ where $\alpha = f(a)$, $\beta = f(b)$. It is easily seen by considering the function $-f(x)$ or the function $f(-x)$.

But if a continuous on $[a, b]$ function $y = f(x)$ is not strictly monotone on $[a, b]$, then we can define for it an inverse, but the latter will be *many-valued* in any case for some y's.

Example. The function

$$y = \sin x, \quad x \in (-\infty, \infty),$$

is continuous, but not monotone. The set of its values y fills the closed interval $[-1, 1]$. To every y from this interval there corresponds an infinite number of values x for which $y = \sin x$.

However, for example, on the closed interval $[-\pi/2, \pi/2]$ the function $y = \sin x$ is continuous and strictly increasing; it has a continuous inverse which is, as we know, designated as

$$x = \arcsin y \quad (-1 \leqslant y \leqslant 1).$$

Sec. 3.7. Uniform Continuity of a Function

Let a function f be continuous on a closed interval $[a, b]$ (or on an open interval, or on a half-interval). Then for every point x_0 of this closed interval (open interval, half-interval), given $\varepsilon > 0$, we can find $\delta > 0$ such that

$$| f(x) - f(x_0) < \varepsilon,$$

as soon as

$$| x - x_0 | < \delta, \quad x \in [a, b] \text{ (or } (a, b), [a, b), (a, b]).$$

With a variable x_0 and constant ε, the number δ, generally speaking, changes: it depends not only on ε, but also on x_0. As is seen from Fig. 29, the number δ which is suitable for the flat part of the graph, may turn out to be too large for its steep portion.

In connection with this, it is natural to separate those continuous functions for which, for a given $\varepsilon > 0$, we can indicate a $\delta > 0$ at once for all x's belonging to the set on which the function is defined.

Let us begin with the definition.

Definition 1. *A function f defined on an arbitrary set X is uniformly continuous on this set if for any $\varepsilon > 0$ there is $\delta > 0$ dependent only on ε such that*

$$|f(x') - f(x'')| < \varepsilon$$

for all x', $x'' \in X$, satisfying the inequality $|x' - x''| < \delta$.

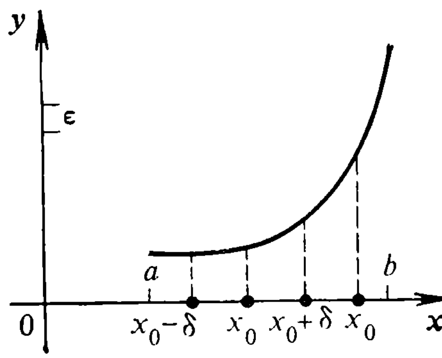

Fig. 29

It is easy to see that if a function is uniformly continuous on a set X, then all the more it is uniformly continuous on any of its subsets X' $(X' \subset X)$. The converse is, generally speaking, not true.

Theorem 1. *If a function f is defined and continuous on a closed interval $[a, b]$, then it is uniformly continuous on this interval.*

Proof. Let us suppose that the statement of the theorem is false. Then there exists $\varepsilon > 0$ such that for any $\delta > 0$ there is a pair of points x', $x'' \in [a, b]$, satisfying the inequality $|x' - x''| < \delta$, for which

$$|f(x') - f(x'')| \geqslant \varepsilon.$$

Let us be given a sequence of positive numbers δ_n $(n = 1, 2, \ldots)$ tending to zero. For every δ_n we can find points x'_n, $x''_n \in [a, b]$ such that

$$|x'_n - x''_n| < \delta_n, \text{ but } |f(x'_n) - f(x''_n)| \geqslant \varepsilon. \quad (1)$$

Since the points of the sequence $\{x'_n\}$ belong to $[a, b]$, this sequence is bounded and, by the Bolzano-Weierstrass theorem, contains a subsequence $\{x_{n_k}\}$ convergent to a certain point $x_0 \in [a, b]$. Since $x'_{n_k} - x''_{n_k} \to 0$, $k \to \infty$, the subsequence $\{x''_{n_k}\}$ also converges to the point x_0. By the hypothesis, the function f is continuous on $[a, b]$ and, consequently, is continuous at the point x_0. Of course, if $x_0 = a$ or $x_0 = b$, then f should be regarded as continuous at x_0 from the right or, respectively, from the left. Therefore

$$\lim_{k\to\infty} f(x'_{n_k}) = \lim_{k\to\infty} f(x''_{n_k}) = f(x_0).$$

On passing to the limit in (1) as $k \to \infty$, we get

$$\varepsilon \leqslant \lim_{k\to\infty} |f(x'_{n_k}) - f(x''_{n_k})| = |f(x_0) - f(x_0)| = 0, \qquad (2)$$

and we have arrived at a contradiction: $\varepsilon \leqslant 0$.

Note that in (2) we took advantage of the continuity of the function $|u|$ (see Sec. 3.3, Example 8). The theorem has been proved.

Example 1. The function

$$y = \sin(1/x)$$

is continuous on a closed interval $[\delta, 1]$, $\forall\, \delta > 0$, therefore, by Theorem 1, it is uniformly continuous on this interval.

On the other hand, although continuous on the half-interval $(0, 1]$, this function is not uniformly continuous on it. This shows that the requirement in Theorem 1 that a continuous function be defined on a closed but not on an open interval is essential.

Let us make sure that our function is not uniformly continuous on $(0, 1]$. The points $x_k = \frac{2}{\pi(2k+1)}$ $(k = 0, 1, 2, \ldots)$, obviously belong to the half-interval $(0, 1]$, and for them

$$|f(x_{k+1}) - f(x_k)| = \left|\sin\frac{\pi(2k+3)}{2} - \sin\frac{\pi(2k+1)}{2}\right|$$
$$= |(-1)^{k+1} - (-1)^k| = 2.$$

If we take $\varepsilon = 1$, then for any $\delta > 0$ there is k such that

$$|x_{k+1} - x_k| = \frac{4}{\pi(2k+3)(2k+1)} < \delta,$$

while

$$|f(x_{k+1}) - f(x_k)| = 2 > \varepsilon = 1.$$

It follows from the aforesaid that our function cannot be extended to the closed interval $[0, 1]$ by determining it at the point $x = 0$ so that it becomes continuous on $[0, 1]$, since then, according to Theorem 1, it would be uniformly continuous on $[0, 1]$, and, consequently, on $(0, 1]$, which is impossible.

Sec. 3.8. Elementary Functions

The functions C (constant), x^n, a^x, $\log_a x$, $\sin x$, $\cos x$, $\tan x$, Arc sin x, Arc cos x, Arc tan x will be called *simplest elementary functions.*

Applying to these functions a finite number of arithmetic operations or operation of superposition (function of function), we shall obtain new, more composite functions which will be called *elementary functions.* For instance, $y = \ln(e^x + \sin^2 x + 1)$ is an elementary function.

Elementary functions are known to us from school mathematics. There much attention was drawn to their properties, while the functions themselves were defined not always rigorously.

Here, it will be useful to discuss these questions from the point of view of general regularities and rules of mathematical analysis already studied by us.

(a) **The Constant Function C.** Here to every real number (point) x there corresponds the value y equal to one and the same number C. The graph of this function (in a rectangular coordinate system) is a straight line parallel to the x-axis passing at a distance $|C|$ from the x-axis and lying above this axis if $C > 0$ and below it if $C < 0$. This is a continuous function throughout the entire real axis (Sec. 3.3, Example 1).

(b) **The Power Function** x^n (n constant). For a natural $n \in \mathbb{N}$ this function is defined throughout the real axis. In order to compute it (theoretically!) we expand x

into a decimal ($x = \pm\alpha_0 \cdot \alpha_1\alpha_2 \ldots$) and multiply this decimal by itself n times, each time applying the rule for multiplying decimals (see Sec. 1.6 (11)) and the rule for signs.

The function x^n is continuous, since it is a product of continuous functions ($y = x$) taken in a finite number. It is strictly increasing on $[0, \infty)$ which is seen from the relations

$$x_2^n - x_1^n = (x_2 - x_1)(x_2^{n-1} + x_2^{n-2}x_1 + \ldots + x_1^{n-1}) > 0$$

for $x_1 < x_2$. Besides, it tends to $+\infty$ as $x \to +\infty$. Indeed, if $x \geqslant 1$, then $x^n = x^{n-1}x \geqslant x$ ($n > 1$) and as $x \to \infty$, $x^n \to \infty$.

Hence, for a natural n the function $\varphi(x) = x^n$ is continuous and strictly increases on $[0, \infty)$ and for it $\varphi(0) = 0$, $\sup\limits_{x\in[0,\infty)} \varphi(x) = +\infty$. But then, by Theorem 1′ proved in Sec. 3.6, the function $y = x^n$ maps the half-interval $X = [0, \infty)$ onto the half-interval $Y = [0, \infty)$ and there exists its inverse which is a one-valued, continuous, strictly increasing function. This function is denoted as

$$x = y^{1/n} = \sqrt[n]{y} \quad (y \geqslant 0)$$

and is called an *arithmetic value of the nth root of y.*

Note that for $y > 1$ ($n \geqslant 1$)

$$\sqrt[n]{y} > \sqrt[n]{1} = 1. \tag{1}$$

We should like to stress that *if a is an arbitrary nonnegative number* ($0 \leqslant a < \infty$), *then for it, on the basis of the aforesaid, there exists a unique nonnegative number* $b = a^{1/n}$ (*the arithmetic value of the* nth *root of* a) *such that* $b^n = a$.

Hence, we have proved the *existence of the nth root of* a ($a \geqslant 0$).

This assertion was absent from the school course of mathematics. This course introduced the notion of the nth root of a ($a \geqslant 0$) and dealt with the properties of the nth roots, but did not prove that such roots exist, since the fact of their existence went without saying.

Note that if $n = 2k + 1$ is an odd number ($k = 0, 1, 2, \ldots$), then $y = x^n$ is an odd function ($(-x)^n = -x^n$). It is continuous, obviously, strictly increases on $(-\infty, \infty)$ and possesses the following properties

$$\inf_{x \in (-\infty, \infty)} x^n = -\infty, \qquad \sup_{x \in (-\infty, \infty)} x^n = +\infty.$$

Therefore, by Theorem 1′ proved in Sec. 3.6, the function $y = x^{2k+1}$ maps $(-\infty, \infty)$ onto $(-\infty, \infty)$ and has on $(-\infty, \infty)$ an inverse which is a continuous, strictly increasing function

$$x = \sqrt[n]{y}, \quad y \in (-\infty, \infty), \quad n = 2k + 1.$$

Here, the expression $\sqrt[n]{y}$ for $y > 0$ is understood as the arithmetic value of the nth root of y, i.e. as a positive number the nth power of which is equal to y. But if $y < 0$, then

$$\sqrt[n]{y} = -\sqrt[n]{|y|},$$

where the root in the right-hand side is understood in arithmetic meaning.

For an even $n = 2k$ ($k = 1, 2, \ldots$) $y = x^n$ is an even continuous function. It maps the open interval $(-\infty, +\infty)$ onto the half-interval $[0, \infty)$. But it is not monotone on $(-\infty, +\infty)$ and its inverse is a two-valued function:

$$x = \pm \sqrt[n]{y} \quad (y \geqslant 0).$$

The value $y = 0$ is the only one for which it is one-valued.

Figures 30 and 31 represent the graphs of some functions x^n, $x^{1/n}$.

In school mathematics the function x^n is also defined for rational n's. Let $p, q \in \mathbf{N}$. It is assumed for $x \geqslant 0$ that

$$x^{p/q} = \sqrt[q]{x^p},$$

and it is proved that

$$x^{p/q} = x^{sp/sq} = (\sqrt[q]{x})^p \quad (x \geqslant 0).$$

It is also assumed that

$$x^{-p/q} = \frac{1}{\sqrt[q]{x^p}} \quad (x > 0)$$

and

$$x^0 = 1.$$

Here, for any rational n the following property characteristic of the power function is proved:

$$(xy)^n = x^n y^n \quad (x, y > 0).$$

Note that the function $y = x^{p/q}$ $(p, q \in \mathbf{N})$, as is easily seen, is continuous and strictly increases on the half-interval $[0, \infty)$ and maps $[0, \infty)$ onto $[0, \infty)$, therefore

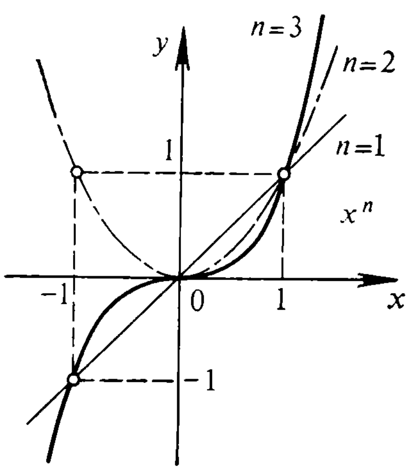

Fig. 30

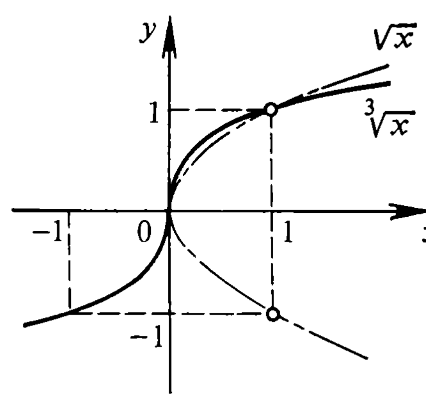

Fig. 31

it has an inverse which is a continuous, strictly increasing function defined, obviously, by the equality

$$x = y^{q/p}, \quad 0 \leqslant y < \infty.$$

As to the function $y = x^{-p/q}$ $(p, q \in \mathbf{N})$, it is strictly decreasing and continuous on the open interval $(0, \infty)$ and maps $(0, \infty)$ onto $(0, \infty)$. Therefore, it has on $(0, \infty)$ an inverse which is a strictly decreasing, continuous function determined by the equality

$$x = y^{-q/p} \quad (y > 0).$$

Obviously,

$$\lim_{\substack{x\to 0\\ x>0}} x^{-p/q} = +\infty, \quad \lim_{\substack{y\to 0\\ y>0}} y^{-q/p} = +\infty.$$

The power function x^n can also be defined (for $x > 0$, and for $n \geqslant 0$ also for $x = 0$) for irrational n's, but this is better done with the aid of the exponential function a^x (see below (c)).

Note that we were interested only in the real roots of the equation $y = x^n$. But if we looked for complex roots, we would find n different roots for any $y \neq 0$ (see below, Sec. 5.3).

Example 1. Let us show that

$$\lim_{n\to\infty} \sqrt[n]{n} = 1. \tag{2}$$

Indeed, by the binomial formula, for $\lambda > 0$ we have

$$(1+\lambda)^n = 1 + n\lambda + \frac{n(n-1)}{2!}\lambda^2 + \ldots > 1 + \frac{n(n-1)}{2}\lambda^2.$$

Setting in this inequality $\lambda = \sqrt[n]{n} - 1$ (>0 for $n \geqslant 2$, see (1)), we get

$$n > 1 + \frac{n(n-1)}{2}(\sqrt[n]{n} - 1)^2$$

or

$$\frac{2}{n} > (\sqrt[n]{n} - 1)^2 > 0 \quad (n \geqslant 2).$$

Taking the square root (arithmetically!), by virtue of strict monotonicity of the function $\sqrt{x}$, we obtain

$$\sqrt{2/n} > \sqrt[n]{n} - 1 > 0$$

or

$$\sqrt{2/n} + 1 > \sqrt[n]{n} > 1 \quad (n \geqslant 2). \tag{3}$$

Finally, taking advantage of the continuity of the function $\sqrt{x}$ at the point $x = 0$, on passing to the limit for $n \to \infty$, from (3) we get equality (2) (see Theorem 5 proved in Sec. 2.1).

(c) **The Exponential Function a^x ($a > 0$, $a \neq 1$).** Hereafter, we shall denote rational numbers by the Greek letters α, β, γ, ... For the time being, let $a > 1$.

The function a^x, where x is rational, is also studied in elementary courses of mathematics (see also (b)).

We also know that:

(1_1) $a^\alpha > 0$,

(2_1) $a^\alpha a^\beta = a^{\alpha+\beta}$,

(3_1) $a^\alpha < a^\beta$ ($\alpha < \beta$, $a > 1$),

(4_1) $(a^\alpha)^\beta = a^{\alpha\beta}$.

Let us add one more:

(5_1) $a^{\alpha_n} \to +\infty$, $\alpha_n \to +\infty$, $a > 1$.

Now we are going to prove Property (5_1). Let us write a in the form $a = 1 + \lambda$ ($\lambda > 0$!). Then

$$a^n = (1 + \lambda)^n = 1 + n\lambda + \ldots > 1 + n\lambda.$$

The right-hand member of this inequality tends to infinity as $n \to \infty$, consequently, so does the left-hand one. Further, regarding $[\alpha_n]$ as the integral part of α_n, we shall have

$$a^{\alpha_n} \geqslant a^{[\alpha_n]},$$

and if $\alpha_n \to +\infty$, then $[\alpha_n] \to +\infty$, consequently, $a^{[\alpha_n]} \to +\infty$. But then $a^{\alpha_n} \to +\infty$.

Let x be an arbitrary rational number. Let us prove that a^x *can be defined as the supremum*

$$\sup_{\alpha<x} a^\alpha = a^x \tag{4}$$

of the numbers a^α extended to all rational numbers $\alpha < x$.

Indeed, by virtue of (3_1),

$$a^\alpha < a^x, \quad \forall\, \alpha < x, \tag{5}$$

i.e. the first property of the supremum is fulfilled. On the other hand, let us construct a strictly increasing sequence of rational numbers α_n tending to x ($\alpha_n \to x$). For it $a^{\alpha_n} \to a^x$, $n \to \infty$. This property is proved below (see (8)). Therefore, for any $\varepsilon > 0$ we can find an n such that

$$a^x - \varepsilon < a^{\alpha_n} < a^x, \tag{6}$$

i.e. the second property of the supremum is fulfilled. From (5) and (6) there follows (4).

Let now x be an arbitrary irrational number, and let m be a natural number greater than x ($m > x$). There takes place an obvious inequality

$$\underset{\alpha<x}{a^{\alpha}} < a^{m},$$

i.e. the set of numbers a^{α} extended to the rational numbers $\alpha < x$ is bounded. But then there exists the supremum of this set

$$\sup_{\alpha<x} a^{\alpha}.$$

This is a quite definite number which will be denoted by a^x:

$$a^x = \sup_{\alpha<x} a^{\alpha}. \tag{7}$$

By this the function a^x is defined for all real x's ($x \in \in (-\infty, \infty)$). It is called the *exponential function*.

Thus, the *function* a^x is defined as the supremum of the numbers a^{α} *extended to all rational numbers* $\alpha < x$.

If x is rational, then this definition coincides with the old definition (yields one and the same number), while for irrational x's it gives new numbers a^x.

It can be proved that the function a^x defined with the aid of equality (7) possesses the following important properties:

(1) $a^x > 0$,
(2) $a^x a^y = a^{x+y}$,
(3) $a^x < a^y$ ($x < y$, $a > 1$),
(4) $a^x \to 1$, $x \to 0$,
(5) $a^x \to \infty$, $x \to \infty$ ($a > 1$),
(6) $a^x \to 0$, $x \to -\infty$ ($a > 1$),
(7) $a^{xy} = (a^x)^y$.

Note that from Properties (2) and (4) there follows the continuity of the function a^x for any value $x_0 \in \in (-\infty, \infty)$:

$$|a^x - a^{x_0}| = |a^{x_0}(a^{x-x_0} - 1)| = a^{x_0}|a^{x-x_0} - 1| \to 0 \tag{8}$$

for $x - x_0 \to 0$.

Henceforward we shall hold that $a > 1$.

The Proof of Property (1). For any x there exists $\alpha_0 < x$, and therefore

$$0 < a^{\alpha_0} < \sup_{\alpha<x} a^{\alpha} = a^x, \quad \text{i.e.} \quad 0 < a^x.$$

The Proof of Property (2). Let $\alpha < x$ and $\beta < y$, then $\alpha + \beta < < x + y$. On the other hand, let γ be an arbitrary rational number satisfying the inequality

$$\gamma < x + y. \tag{9}$$

Let us show that γ can be written in the form

$$\gamma = \alpha + \beta, \quad \text{where} \quad \alpha < x,\ \beta < y. \tag{10}$$

It follows from (9) that

$$\gamma - y < x.$$

Now let us choose a rational α for which

$$\gamma - y < \alpha < x, \tag{11}$$

and make

$$\beta = \gamma - \alpha. \tag{12}$$

Then from the first inequality of (11) it follows that

$$\beta < y. \tag{13}$$

And so, the set of all sums $\alpha + \beta$, where $\alpha < x$, $\beta < y$, is equal to the set of all $\gamma < x + y$:

$$\{\alpha+\beta\}_{\substack{\alpha<x\\ \beta<y}} = \{\gamma\}_{\gamma<x+y}.$$

Therefore

$$a^{x+y} = \sup_{\gamma<x+y} a^{\gamma} = \sup_{\substack{\alpha<x\\ \beta<y}} a^{\alpha+\beta} = \sup_{\substack{\alpha<x\\ \beta<y}} (a^{\alpha}a^{\beta}) = \sup_{\alpha<x} a^{\alpha} \sup_{\beta<y} a^{\beta} = a^x \cdot a^y$$

(see Sec. 2.8, Problem 2).

The Proof of Property (3). Let $x < y$ and β_1 and β_2 be some rational numbers for which $x < \beta_1 < \beta_2 < y$. Then

$$a^x = \sup_{\alpha<x} a^{\alpha} \leqslant \sup_{\alpha<\beta_1} a^{\alpha} = a^{\beta_1} < a^{\beta_2} \leqslant \sup_{\beta<y} a^{\beta} = a^y,$$

and we have proved that

$$a^x < a^y.$$

The Proof of Property (4). For a natural $n > a$ ($a > 1$!)

$$1 < a^{1/n} < n^{1/n} \underset{n\to\infty}{\to} 1$$

(see Example 1, (b)) and we have proved for the time being that

$$\lim_{n\to\infty} a^{1/n} = 1. \tag{14}$$

Now let there by given an arbitrary sequence of positive numbers $x_n < 1$, tending to zero ($x_n \to 0$, $x_n > 0$). Let $k_n = [1/x_n]$ be the integral part of $1/x_n$. Then $0 < x_n < 1/k_n$ and

$$1 = a^0 < a^{x_n} \leqslant a^{1/k_n}.$$

Therefore, by virtue of (14)

$$\lim_{n\to\infty} a^{x_n} = 1.$$

By virtue of the arbitrariness of the sequence $\{x_n\}$, where $x_n > 0$, this proves that there exists the right-hand limit

$$\lim_{\substack{x\to 0\\ x>0}} a^x = 1, \tag{15}$$

but then there also exists the left-hand limit

$$\lim_{\substack{x\to 0\\ x<0}} a^x = \lim_{\substack{-x\to 0\\ -x>0}} \frac{1}{a^{-x}} = \frac{1}{\lim\limits_{\substack{u\to 0\\ u>0}} a^u} = \frac{1}{1} = 1. \tag{16}$$

It follows from (15) and (16) that

$$\lim_{x\to 0} a^x = 1$$

(see Sec. 3.2, (6), (7)). This completely proves Property (4).

The Proof of Property (5). Let us take an arbitrarily large number $M > 0$. There exists a rational number α such that $a^\alpha > M$, therefore

$$M < a^\alpha < a^x, \quad \forall x > \alpha.$$

Consequently, $a^x \to +\infty$, as $x \to +\infty$.

The Proof of Property (6).

$$\lim_{x\to -\infty} a^x = \lim_{-x\to +\infty} \frac{1}{a^{-x}} = \frac{1}{\lim\limits_{u\to +\infty} a^u} = 0.$$

The Proof of Property (7). Note that for a natural m, by (2), we have

$$a^{xm} = a^x \dots a^x = (a^x)^m; \quad \left(a^{\frac{x}{m}}\right)^m = a^{\frac{x}{m}} \dots a^{\frac{x}{m}} = \left(a^{\frac{x}{m}}\right)^m = a^x;$$
$$a^{\frac{x}{m}} = (a^x)^{\frac{1}{m}}.$$

For a rational number $\frac{p}{q} > 0$

$$(a^x)^{\frac{p}{q}} = \left((a^x)^{\frac{1}{q}}\right)^p = \left(a^{\frac{x}{q}}\right)^p = a^{x\frac{p}{q}}.$$

Further, if y is an arbitrary positive number and $\alpha_n \to y$, where α_n are rational numbers, then, by virtue of the continuity of expo-

nential function,

$$a^{xy} = \lim_{n\to\infty} a^{x\alpha_n} = \lim_{n\to\infty} (a^x)^{\alpha_n} = (a^x)^y$$

and we have proved (7) for $y > 0$. If $y < 0$, then $(y = -|y|)$

$$a^{xy} = a^{-x|y|} = \frac{1}{a^{x|y|}} = \frac{1}{(a^x)^{|y|}} = (a^x)^{-|y|} = (a^x)^y.$$

If $0 < a < 1$, then we set

$$a^x = \frac{1}{(1/a)^x}.$$

Properties (1), (2), (4) then remain unchanged. Property (3) has the form: $a^x > a^y$ $(x < y)$.

Property (5): $a^x \to 0$, $x \to +\infty$.

And Property (6) has the form: $a^x \to +\infty$, $x \to -\infty$.

(d) The Function $\log_a x$. Let us assume that $a > 1$. Since the function $y = a^x$ is continuous and strictly increasing on the interval $(-\infty, \infty)$ and maps the interval $(-\infty, \infty)$ onto the interval $(0, \infty)$, it has an inverse which is continuous and strictly increases on $(0, \infty)$. The inverse function is called the *logarithm y to the base a* and is denoted by the symbol

$$x = \log_a y, \quad y \in (0, \infty).$$

It follows from the aforesaid that (we replace y by x)

$$\lim_{x\to+\infty} \log_a x = +\infty, \quad \lim_{\substack{x\to 0\\ x>0}} \log_a x = -\infty.$$

For $a < 1$ we reason in a similar way. The function a^x also maps the real axis $(-\infty, \infty)$ onto the semiaxis $(0, \infty)$ but strictly decreasing. Its inverse $\log_a x$ defined on $(0, \infty)$ will also strictly decrease and now

$$\lim_{x\to+\infty} \log_a x = -\infty, \quad \lim_{\substack{x\to 0\\ x>0}} \log_a x = +\infty.$$

The following identities are valid (see Sec. 3.6):

$$a^{\log_a x} = x \quad (0 < x < +\infty), \quad \log_a a^x = x \quad (-\infty < x < +\infty) \tag{17}$$

$(a \neq 1)$. Hence, by virtue of the properties of the function a^x, for $x, y > 0$ we have

$$a^{\log_a (xy)} = xy = a^{\log_a x} a^{\log_a y} = a^{\log_a x + \log_a y}$$

and

$$\log_a (xy) = \log_a x + \log_a y.$$

Replacing x by x/y in this equality, we get

$$\log_a x - \log_a y = \log_a (x/y).$$

Further (see (7))

$$a^{\log_a x^y} = x^y = (a^{\log_a x})^y = a^{y \log_a x} \quad (x > 0),$$

therefore

$$\log_a x^y = y \log_a x \quad (a \neq 1,\ x > 0). \tag{18}$$

Finally, we should like to note that for positive numbers a and b different from 1 the following takes place:

$$a^{\log_a b \cdot \log_b a} = (a^{\log_a b})^{\log_b a} = b^{\log_b a} = a,$$

and, consequently,

$$\log_a b \log_b a = 1.$$

The logarithm of a number a to the base e is called a *natural logarithm* of the number a and is denoted as follows: $\log_e a = \ln a$.

(e) Let us return to the power function

$$y = x^n, \quad 0 < x < \infty.$$

After the aforesaid we may assert that this function makes sense not only for rational n's, but also for irrational n's. It can also be written in the following manner (see (17) and (18)):

$$x^n = e^{n \ln x}, \tag{19}$$

whence it is seen that it is continuous as a superposition of continuous functions.

For $n > 0$ the function strictly increases and possesses the following properties:

$$\lim_{\substack{x \to 0 \\ x > 0}} x^n = 0, \quad \lim_{x \to +\infty} x^n = +\infty.$$

For $n > 0$ it is natural to hold that $0^n = 0$, then the function becomes continuous from the right at the point $x = 0$.

For $n < 0$ the function x^n is continuous, strictly decreasing on the positive semiaxis and possessing the following properties:

$$\lim_{\substack{x \to 0 \\ x > 0}} x^n = +\infty, \quad \lim_{x \to +\infty} x^n = 0.$$

From formula (19) there follows the characteristic property of the power function

$$(xy)^n = e^{n \ln (xy)} = e^{n \ln x} e^{n \ln y} = x^n y^n$$

$(x,\ y > 0)$.

(f) **The Function $y = u\,(x)^{v(x)}$.** Let the functions $u\,(x)$ and $v\,(x)$ be given in a neighbourhood of a point a except, possibly, this point itself, $u\,(x) > 0$ and $\lim\limits_{x \to a} u\,(x) = A > 0$, $\lim\limits_{x \to a} v\,(x) = B$ (A and B finite numbers). Then

$$\lim_{x \to a} u\,(x)^{v(x)} = A^B. \tag{20}$$

Indeed (see (17), (18))

$$\lim_{x \to a} u\,(x)^{v(x)} = \lim_{x \to a} e^{v(x) \ln u(x)} = e^{\lim\limits_{x \to a} [v(x) \ln u(x)]}$$

$$= e^{\lim\limits_{x \to a} v(x) \lim\limits_{x \to a} \ln u(x)} = e^{B \ln A} = A^B.$$

The second equality of this "chain" uses the continuity of the function e^z, while the fourth equality uses the continuity of the function $\ln z$.

If $u\,(x)$ and $v\,(x)$ are continuous at the point $x = a$ and $u\,(a) > 0$, then in a certain neighbourhood of this point $u\,(x) > 0$ (see Sec. 3.3, Theorem 4) and $A = u\,(a)$, $B = v\,(a)$. Therefore, by (20),

$$\lim_{x \to a} u\,(x)^{v(x)} = u\,(a)^{v(a)}.$$

Let us mention some interesting cases not stipulated by equality (20), when (for $x \to a$, $u > 0$) $u \to +\infty$, $v \to 0$; $v \to \infty$, $u \to 1$; $v \to 0$, $u \to 0$.

In these cases the theorem on the limit of $v \ln u$ does not work. It is impossible to give the formula for $\lim\limits_{x \to a} u^v$ in advance without having more precise information concerning the character of tending of u and v to the indicated limits. These cases yield indeterminate forms ∞^0, 1^∞, 0^0 for the expression u^v in the neighbourhood of the point a.

(g) The Trigonometric Functions. The functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$ are known to the reader from trigonometry where they are defined on the basis of geometrical considerations. We shall also use these definitions.

We could give a purely analytical definition of the trigonometric functions, but we are not intended to do this.

Let us note that the function $y = \sin x$ is continuous (see Sec. 3.3, Example 3) and strictly increases on the closed interval $[-\pi/2, \pi/2]$, mapping this interval onto the closed interval $[-1, 1]$. But then it has an inverse one-valued continuous function

$$x = \text{arc sin}\, y, \quad y \in [-1, 1].$$

But the function $y = \sin x$, being considered on the entire axis $(-\infty, \infty)$, has a many-valued, even infinite-valued, inverse function $\text{Arc sin}\, y$, all the values of which are computed by the formula

$$x = \text{Arc sin}\, y = (-1)^k \,\text{arc sin}\, y + k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots), \tag{21}$$

i.e. to every $y \in [-1, 1]$ there corresponds the set e_y of values of x determined by formula (21).

Similarly, inverses of the functions

$$y = \cos x, \quad 0 \leqslant x \leqslant \pi,$$

$$y = \tan x, \quad -\pi/2 < x < \pi/2,$$

will be the functions

$$x = \text{arc cos}\, y, \quad y \in [-1, 1],$$

$$x = \text{arc tan}\, y, \quad y \in (-\infty, \infty),$$

and for the same functions considered on the real axis the respective inverse functions have the following form:

$$x = \mathrm{Arc}\cos y = \pm\arccos y + 2k\pi,$$
$$x = \mathrm{Arc}\tan y = \arctan y + k\pi$$

$(k = 0, \pm 1, \pm 2, \ldots)$.

(h) The Hyperbolic Functions. The functions

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}$$

are respectively called *hyperbolic sine of* x, *hyperbolic cosine of* x, *hyperbolic tangent of* x, and *hyperbolic cotangent*

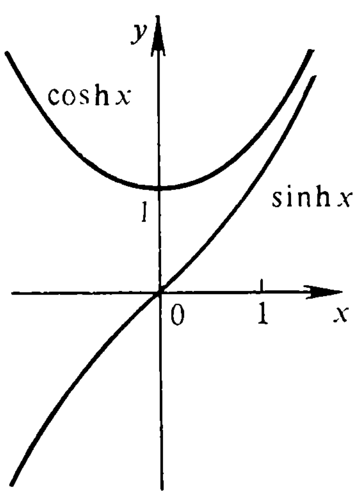

Fig. 32

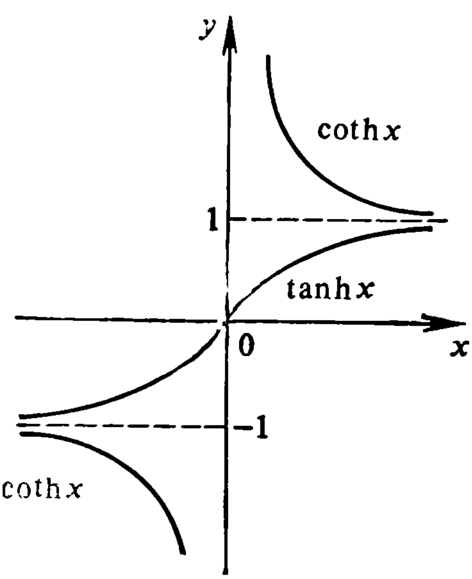

Fig. 33

of x. Their graphs are drawn in Figs. 32 and 33. The functions $\sinh x$, $\cosh x$, and $\tanh x$ are defined on $(-\infty, \infty)$, and the function $\coth x$ on the same interval, except for the point $x = 0$.

It is easy to check that for these functions there take place formulas similar to those from standard trigonometry (but not always coincident!). For instance,

$$\sinh(x + y) = \sinh x \cosh y + \sinh y \cosh x,$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y.$$

Setting $y = -x$ in the last equality, we get

$$\cosh^2 x - \sinh^2 x = 1.$$

Note that all the functions considered here are continuous in their domains of definition. For sinh x and tanh x there exist inverse functions: *inverse hyperbolic sine* (written $x = \operatorname{Arsinh} y$) and *inverse hyperbolic tangent* (written $x = \operatorname{Artanh} y$) which are one-valued (or one-to-one). The inverse of $\cosh x$ is also one-valued for $x \geqslant 0$.

Sec. 3.9. Remarkable Limits

Theorem 1.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Proof. Since the function $y = \sin x$ is continuous, $\sin x \to \sin 0 = 0$ as $x \to 0$. Therefore the expression $\frac{\sin x}{x}$ represents an indeterminate form of the type $\frac{0}{0}$. Let us evaluate it. From the definition of trigonometric functions and geometrical considerations we have (see Fig. 34)

$$0 < \sin x < x < \tan x$$

for $0 < x < \pi/2$ ($MN = \sin x$, $AM \perp OM$, $AM = \tan x$, $OM = 1$). Hence, dividing by $\sin x > 0$, we get

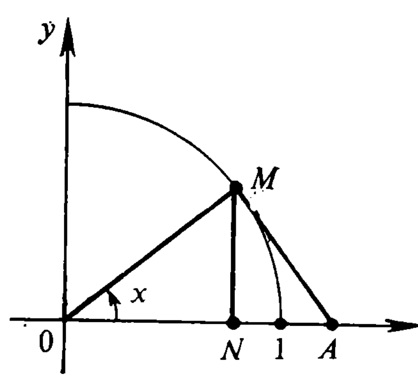

Fig. 34

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x} \quad \text{or} \quad 1 > \frac{\sin x}{x} > \cos x. \tag{1}$$

Inequalities (1) are also true for $-\pi/2 < x < 0$, since the functions $\cos x$ and $\frac{\sin x}{x}$ are even. Furthermore, the function $\cos x$ is continuous, therefore

$$\lim_{x \to 0} \cos x = \cos 0 = 1,$$

and, consequently, passing to the limit in (1), by virtue of Theorem 4 proved in Sec. 3.2, we obtain

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Example 1.

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}$$

$$=\frac{1}{2}\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}\left(\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}\cdot 1=\frac{1}{2}.$$

Theorem 2.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e. \tag{2}$$

Proof. By virtue of definition of the limit of a function, we must show that

$$\left(1+\frac{1}{x_n}\right)^{x_n}\to e,\quad \forall x_n\to\infty. \tag{3}$$

If $x_n = n$ is natural, then this is proved. In order to prove (2), it is sufficient to make sure that (2) is true in two cases: when $x_n \to +\infty$ and when $x_n \to -\infty$, running not necessarily through integral values (see the remark at the end of Sec. 3.2).

Let x_n be an arbitrary variable tending to $+\infty$ ($x_n \to +\infty$), and let $[x_n] = k_n$ be the integral part of the number x_n. Then $k_n \leqslant x_n < k_n + 1 \leqslant x_n + 1 < k_n + 2$ and

$$\left(1+\frac{1}{k_n+1}\right)^{k_n+1}<\left(1+\frac{1}{x_n}\right)^{x_n+1}$$

$$<\left(1+\frac{1}{k_n}\right)^{k_n+2}<e\left(1+\frac{1}{k_n}\right)^2.$$

For $x_n \to +\infty$ $[x_n] = k_n \to +\infty$, whence the first and the last terms of the chain of the inequalities tend to e.

Therefore

$$\left(1+\frac{1}{x_n}\right)^{x_n+1} \to e,$$

and since in this case $1+\dfrac{1}{x_n} \to 1$, we have proved (3) for $x_n \to +\infty$.

If now $x_n \to -\infty$, then $x'_n = -x_n \to +\infty$ and

$$\lim_{x_n \to -\infty}\left(1+\frac{1}{x_n}\right)^{x_n} = \lim_{x'_n \to +\infty}\left(1-\frac{1}{x'_n}\right)^{-x'_n}$$

$$= \lim_{x'_n \to +\infty}\left(\frac{x'_n}{x'_n-1}\right)^{x'_n}$$

$$= \lim_{x'_n \to +\infty}\left[\left(1+\frac{1}{x'_n-1}\right)^{x'_n-1}\left(1+\frac{1}{x'_n-1}\right)\right] = e,$$

i.e. (2) has been proved.

Example 2.

$$\lim_{u \to 0}(1+u)^{1/u} = e.$$

This is obtained from (2) by introducing the substitute $1/x = u$.

Example 3.

$$\lim_{x \to \infty}\left(1+\frac{\alpha}{x}\right)^x = e^\alpha, \quad \lim_{u \to 0}(1+\alpha u)^{1/u} = e^\alpha, \quad \forall\alpha.$$

For $\alpha = 0$ this expression is reduced to the limit $\lim\limits_{x \to \infty} 1^x = 1 = e^0$, since, by definition, $1^x = 1$.

Let now $\alpha \neq 0$. If $x \to \infty$, then $\dfrac{x}{\alpha} \to \infty$ and

$$\left(1+\frac{\alpha}{x}\right)^x = \left[\left(1+\frac{\alpha}{x}\right)^{x/\alpha}\right]^\alpha = u^\alpha \underset{u \to e}{\longrightarrow} e^\alpha.$$

It should be taken into account that the power function u^α is continuous at the point $u = e$ (see Sec. 3.8, (e)).

Example 4.

$$\lim_{x \to 0}\frac{\ln(1+x)}{x} = 1, \quad \lim_{x \to 0}\frac{\log_a(1+x)}{x} = \log_a e = \frac{1}{\ln a}.$$

Since $\ln u$ is a continuous function on $(0, \infty)$, we have (see Example 2)

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \ln(1+x)^{1/x} = \ln \lim_{x\to 0} (1+x)^{1/x} = \ln e = 1.$$

Example 5.

$$\lim_{x\to 0} \frac{a^x-1}{x} = \ln a \quad (a>0) \quad \lim_{x\to 0} \frac{e^x-1}{x} = 1.$$

Indeed, let us set $a^x - 1 = u$. By virtue of the continuity of the exponential function, $u \to 0$ as $x \to 0$. Further, $x \ln a = \ln(1+u)$, therefore (see Example 4)

$$\lim_{x\to 0} \frac{a^x-1}{x} = \lim_{u\to 0} \frac{u}{\ln(1+u)} \ln a = \ln a \cdot \lim_{u\to 0} \frac{u}{\ln(1+u)} = \ln a.$$

Sec. 3.10. Order of a Variable. Equivalence

We shall consider two functions $\varphi(x)$ and $\psi(x)$ defined in a certain neighbourhood $U(a)$ of the point a, except, possibly, for the point a itself. Point a may be finite or infinite ($+\infty$, $-\infty$, or ∞). We shall also regard that $\psi(x) \neq 0$ on $U(a)$. If

$$\lim_{x\to a} \frac{\varphi(x)}{\psi(x)} = 0, \tag{1}$$

then we shall write this fact as follows:

$$\varphi(x) = o(\psi(x)), \quad x \to a, \tag{1'}$$

and say that $\varphi(x)$ *is of higher order (of smallness) than* $\psi(x)$ *as* $x \to a$.

For instance:

$$\begin{aligned}
x^2 &= o(x), & & x \to 0;\\
x^n &= o(x^m), & & x \to 0, \quad \text{if } m < n;\\
x^n &= o(x^m), & & x \to \infty, \quad \text{if } m > n;\\
(x-a)^4 &= o((x-a)^3), & & x \to a;\\
1-\cos x &= o(x), & & x \to 0, \quad \text{since } \lim_{x\to 0} \frac{1-\cos x}{x} = 0.
\end{aligned}$$

The expression $o\,(1)$, $x \to a$, denotes the infinitesimal as $x \to a$, that is, a certain function $\varphi\,(x)$ which tends to zero as $x \to a$ ($\varphi\,(x) \to 0$, $x \to a$). For instance, $\varphi\,(x) = \frac{1}{\ln x} = o\,(1)$, $x \to +\infty$.

Property (1), obviously, expresses the fact that the function $\varphi\,(x)$ can be written in the form $\varphi\,(x) = \varepsilon\,(x)\,\psi\,(x)$ where $\varepsilon\,(x) \to 0$ as $x \to a$.

If the functions φ and ψ related by (1) (or, which is the same, by (1')) are infinitesimals as $x \to a$, then, using the old terminology we also say that $\varphi\,(x)$, as $x \to a$, *is of higher order (of smallness) than the infinitesimal* $\psi\,(x)$. And if φ and ψ in (1) are infinitely large as $x \to a$, then we say that $\varphi\,(x)$ *is an infinitely large quantity of lower order than* $\psi\,(x)$ as $x \to a$, or that $\psi\,(x)$ *is an infinitely large quantity of higher order than* $\varphi\,(x)$ as $x \to a$.

We shall also write

$$\varphi\,(x) \approx \psi\,(x), \quad x \to a \tag{2}$$

and call the functions $\varphi\,(x)$ and $\psi\,(x)$ *equivalent* (*asymptotically equal*) *for* $x \to a$ if the following property is fulfilled:

$$\lim_{x \to a} \frac{\varphi\,(x)}{\psi\,(x)} = 1. \tag{2'}$$

For example,

$$\sin x \approx x, \quad x \to 0, \tag{3}$$

Given below are some more examples (see also examples 1, 4, 5 in Sec. 3.9):

$$1 - \cos x \approx x^2/2, \quad x \to 0, \tag{4}$$

$$\ln\,(1 + x) \approx x, \quad x \to 0, \tag{5}$$

$$e^x - 1 \approx x, \quad x \to 0, \tag{6}$$

$$a^x - 1 \approx x \ln a, \quad x \to 0. \tag{7}$$

Note that if

$$\lim_{x \to a} f\,(x) = A \neq 0,$$

then it is equivalent to the fact that

$$\lim_{x\to a} \frac{f(x)}{A} = 1,$$

which we also agreed to denote in the following manner:

$$f(x) \approx A, \quad x \to a \quad (A \neq 0). \tag{8}$$

The terminology introduced here is needed to simplify computations as also to write the relevant formulas in a briefer notation. It is important here to learn a few simple properties of asymptotically equal (equivalent) functions that are expressed in the theorems below.

Theorem 1. *If*

$$\varphi(x) \approx \psi(x), \quad x \to a, \tag{9}$$

then

$$\psi(x) \approx \varphi(x), \quad x \to a. \tag{10}$$

Proof. The point is that if $\psi(x) \neq 0$ on $U(a)$ and (9) is fulfilled, then, obviously, $\varphi(x)$ is also not equal to zero, possibly, on somewhat reduced neighbourhood. But then

$$\lim_{x\to a} \frac{\psi(x)}{\varphi(x)} = \lim_{x\to a} \frac{1}{\frac{\varphi(x)}{\psi(x)}} = \frac{1}{1} = 1.$$

Theorem 2. *For property* (9) *to be fulfilled, it is necessary and sufficient that*

$$\varphi(x) = \psi(x) + o(\psi(x)), \quad x \to a. \tag{11}$$

Equality (11) should be read as follows: the term added to $\psi(x)$ in order to obtain $\varphi(x)$ possesses the following property: if it is divided by $\psi(x)$, then the obtained quotient will tend to zero if x is made to tend to a.

Proof. Let (9) be valid. Then

$$\frac{\varphi(x)}{\psi(x)} = 1 + \varepsilon(x), \quad \text{where } \varepsilon(x) \to 0 \text{ as } x \to a.$$

Consequently,

$$\varphi(x) = \psi(x) + \varepsilon(x)\psi(x) = \psi(x) + o(\psi(x)), \quad x \to a,$$

and we have proved (11).

Conversely, if (11) is true, then

$$\varphi(x) = \psi(x) + o(\psi(x)) = \psi(x) + \varepsilon(x)\psi(x),$$

where $\varepsilon(x) \to 0$ as $x \to a$. Therefore

$$\frac{\varphi(x)}{\psi(x)} = 1 + \varepsilon(x) \underset{x\to a}{\to} 1,$$

and we have obtained (9).

Theorem 3. *If*

$$\psi(x) \approx \psi_1(x), \quad x \to a,$$

then

$$\lim_{x\to a} [\varphi(x)\psi(x)] = \lim_{x\to a} [\varphi(x)\psi_1(x)], \tag{12}$$

$$\lim_{x\to a} \frac{\varphi(x)}{\psi(x)} = \lim_{x\to a} \frac{\varphi(x)}{\psi_1(x)}. \tag{13}$$

These equalities should be understood in the sense that if there exists in them a right-hand limit, then there also exists a left-hand limit, and they are equal, and conversely.

Hence it follows that if one of these limits does not exist, then the second does not exist either.

Proof. Let us set forth only one of these arguments. Let there exist a right-hand limit in (12). Then

$$\lim_{x\to a} [\varphi(x)\psi(x)] = \lim_{x\to a} \left[\varphi(x)\psi_1(x)\frac{\psi(x)}{\psi_1(x)}\right]$$

$$= \lim_{x\to a} [\varphi(x)\psi_1(x)] \lim_{x\to a} \frac{\psi(x)}{\psi_1(x)} = \lim_{x\to a} [\varphi(x)\psi_1(x)]\cdot 1$$

$$= \lim_{x\to a} [\varphi(x)\psi_1(x)].$$

Example 1. $\tan x \approx x$, $x \to 0$, since

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right) = 1.$$

Example 2.

$$\lim_{x\to 0} \frac{\tan^3 x}{x^3+x} = \lim_{x\to 0} \frac{x^3}{x^3+x} = \lim_{x\to 0} \frac{x^2}{x^2+1} = \frac{0}{1} = 0.$$

Definition. *If for a function $\varphi(x)$ there can be chosen numbers A and m, where $A \neq 0$ such that*

$$\varphi(x) \approx A(x-a)^m, \quad x \to a,$$

then the function $A(x-a)^m$ is said to be the principal (power) asymptotic term of the function $\varphi(x)$ in a neighbourhood of the point a.

The right-hand members of relationships (3)-(7) are, obviously, the principal (asymptotic) terms of their left-hand members for $x \to 0$.

We shall say that f on a set E has the *order* φ or f is of lower order (of smallness) than φ on E, and write

$$f(x) = O(\varphi(x)) \text{ on } E, \tag{14}$$

if

$$|f(x)| \leqslant C\,|\varphi(x)|, \quad \forall x \in E,$$

where C is a positive constant independent of x.

In particular, the equality

$$f(x) = O(1) \text{ on } E$$

denotes the fact that f is bounded on E.

Examples.

(1) $\sin x = O(1)$, $\sin x = O(x)$ on $(-\infty, \infty)$;

(2) $x = O(x^2)$ on $[1, \infty)$;

(3) $x^2 = O(x)$ on $[0, 1]$.

CHAPTER 4

DIFFERENTIAL CALCULUS. FUNCTIONS OF ONE VARIABLE

Sec. 4.1. The Derivative Defined

The notion of the derivative, alongside with the notion of the integral, is the most important in mathematical analysis.

The derivative of a function f at a point x is defined as the limit to which tends the ratio of the increment Δy of the function at that point to the corresponding increment Δx of the argument as Δx tends to zero.

The derivative is usually denoted in the following way:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}. \qquad (1)$$

But other notations such as y', $\frac{df(x)}{dx}$, $\frac{dy}{dx}$ are also widely used. The convenience of this or that notation will be appreciated afterwards by the reader himself.

For any fixed x the quantity $\frac{\Delta y}{\Delta x}$ is a function of Δx:

$$\psi(\Delta x) = \frac{\Delta y}{\Delta x} \quad (\Delta x \neq 0).$$

For the derivative of f at a point x to exist, it is necessary that the function f be defined in a certain neighbourhood of the point x including the point x itself. Then the function $\psi(\Delta x)$ is defined for all sufficiently small Δx different from zero, i.e. for Δx satisfying an inequality $0 < |\Delta x| < \delta$ where δ is a sufficiently small positive number.

Of course, limit (1) exists not for any function f defined in a neighbourhood of the point x. Usually, when we say that a function f has a derivative $f'(x)$ at a point x, it is implied that this derivative is finite, i.e. limit (1)

is finite. But it may occur that there exists an infinite limit (1) equal to $+\infty$, $-\infty$, or ∞. In such cases it is useful to say that the function f has at point x an *infinite derivative* (equal to $+\infty$, $-\infty$, or ∞).

If in formula (1) it is assumed that Δx tends to zero attaining only positive values ($\Delta x > 0$), then the corresponding limit (whenever it exists) is called the *derivative on the right of a function f at a point x*. We shall denote it as $f'(x^+)$.

Analogously, limit (1), when Δx tends to zero running through negative values ($\Delta x < 0$), is termed the *derivative on the left of f at x* (denoted: $f'(x^-)$).

Of course, to compute $f'(x^+)$ (or $f'(x^-)$) it is only necessary that the function f be defined at the point x and on the right of it in a certain neighbourhood (or at x and on the left of x).

Typical in this respect is the case when f is defined on a closed interval $[a, b]$ and has a derivative at all interior points of this interval, i.e. at the points of the open interval (a, b); as to the end points, it has a right derivative at a, and a left derivative at b. In such cases we say that the *function f has a derivative at all the points of the closed interval* $[a, b]$ without stipulating that it has, in fact, only a right derivative at a and a left derivative at b.

It is easy to see that if a function f has both a right and left derivatives at a point x and they are equal to each other, then the function f is said to have a *derivative* at x:

$$f'(x^+) = f'(x^-) = f'(x).$$

But if a right and left derivatives exist at x and they are not equal to each other ($f'(x^+) \neq f'(x^-)$), then the function *f fails to have a derivative at x.*

Example. Consider the function $y = |x|$ (see Fig. 35). For it

$$\frac{\Delta y}{\Delta x} = \frac{|x+\Delta x| - |x|}{\Delta x}.$$

If $x > 0$, then $x + \Delta x > 0$ for sufficiently small Δx and

$$\frac{\Delta y}{\Delta x} = \frac{x+\Delta x - x}{\Delta x} = \frac{\Delta x}{\Delta x} = 1 \quad (x > 0).$$

If $x < 0$, then $x + \Delta x < 0$ for sufficiently small Δx and

$$\frac{\Delta y}{\Delta x} = \frac{-(x+\Delta x)-(-x)}{\Delta x} = -\frac{\Delta x}{\Delta x} = -1 \quad (x<0).$$

Hence,

$$|x|' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \begin{cases} 1, \text{ if } x>0, \\ -1, \text{ if } x<0. \end{cases}$$

Let now $x = 0$. Then

$$\frac{\Delta y}{\Delta x} = \frac{|\Delta x|}{\Delta x} = \operatorname{sgn} \Delta x \, \frac{\Delta x}{\Delta x} = \operatorname{sgn} \Delta x = \begin{cases} 1, \text{ if } \Delta x>0, \\ -1, \text{ if } \Delta x<0. \end{cases}$$

Therefore

$$\lim_{\substack{\Delta x \to 0 \\ \Delta x>0}} \frac{\Delta y}{\Delta x} = 1, \quad \lim_{\substack{\Delta x \to 0 \\ \Delta x<0}} \frac{\Delta y}{\Delta x} = -1.$$

Thus, at the point $x = 0$ the function $|x|$ has a right derivative equal to 1 and a left derivative equal to -1.

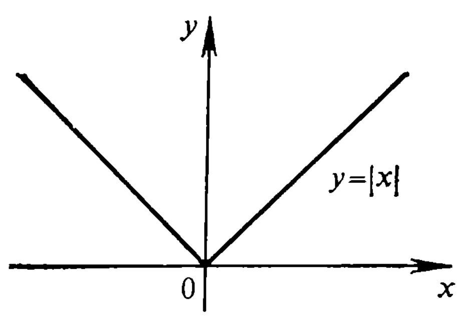

Fig. 35

This means that the function $|x|$ has no *derivative* at the point $x = 0$.

We know (see Sec. 3.3, Example 8) that $|x|$ is a continuous function for all values of x, including $x = 0$; therefore it may serve as an example of a function which is continuous everywhere, but has no derivative at a certain point. There are functions which are continuous throughout the entire real axis and have no derivative at some points of the axis. One of them is the Weierstrass function

representable by the trigonometric series

$$\mu(x)=\sum_{n=0}^{\infty} 2^{-n}\sin 2^{n}\pi x.$$

However, we do not intend to go into further details.

On the other hand, *any function, having a finite derivative at a point x, is continuous at this point.*

Indeed, let limit (1) exist at a point x and be finite. This fact can be written in the following manner:

$$\frac{\Delta y}{\Delta x}=f'(x)+\varepsilon(\Delta x), \tag{2}$$

where $\varepsilon(\Delta x)\to 0$ for $\Delta x\to 0$, i.e. $\varepsilon(\Delta x)$ is an infinitesimal as $\Delta x\to 0$. It follows from (2) that

$$\Delta y = f'(x)\,\Delta x+\Delta x\cdot\varepsilon(\Delta x).$$

Passing in this equality to the limit as $\Delta x\to 0$, we obtain

$$\lim_{\Delta x\to 0}\Delta y=0,$$

which shows that f is continuous at x.

Let us consider some important applications of the derivative.

Instantaneous Velocity. Let the function $s = f(t)$ express the law of motion of a point in a straight line. Here s denotes the distance (taking into account the sign) of the point to the initial point 0 at an instant t. The path covered by the point during the time interval $[t, t+\Delta t]$ is equal to

$$\Delta s = f(t+\Delta t)-f(t).$$

Its average velocity during this time interval is

$$v_{\text{av}}=\frac{\Delta s}{\Delta t}.$$

And its *true* (*instantaneous*) *velocity* at the instant t is, naturally, defined as the limit

$$v=\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}=f'(t).$$

Current Intensity. Let $Q = f(t)$ be the quantity of electricity passing through the cross section of a conductor during time t. Then

$$\frac{\Delta Q}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

is an average current intensity during the time interval $[t, t + \Delta t]$. And the limit

$$\lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = Q' = I$$

is the current intensity at the moment t.

Mass Density. Let (Fig. 36) a homogeneous mass be

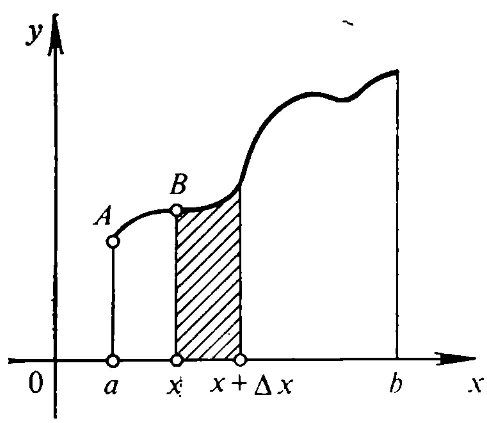

Fig. 36

distributed nonuniformly along the interval $[a, b]$ of the x-axis so that the mass loaded on the interval $[a, x]$ is equal to

$$M = F(x) \quad (a \leqslant x \leqslant b).$$

This quantity is proportional to the area of the figure $aABx$. Hence, M is a function of x $(M = F(x))$. The mass distributed along the interval $[x, x + \Delta x]$ is, obviously, equal to

$$\Delta F = F(x + \Delta x) - F(x).$$

Its mean (average) density on this interval equals $\frac{\Delta F}{\Delta x}$ and the limit

$$\lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = F'(x) = \mu$$

is the *true mass density at the point* x.

Sec. 4.2. Geometrical Meaning of the Derivative

Let a continuous function $y = f(x)$ be defined on an open interval (a, b). Its graph is called a *continuous curve*. We shall denote it by Γ. Let us take a point $A = (x, f(x))$ on Γ (Figs. 37 and 38) and aim at determining the tangent at this point. For this purpose, let

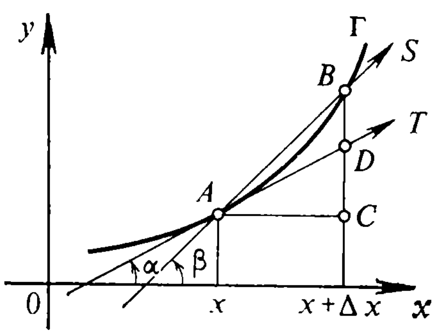

Fig. 37

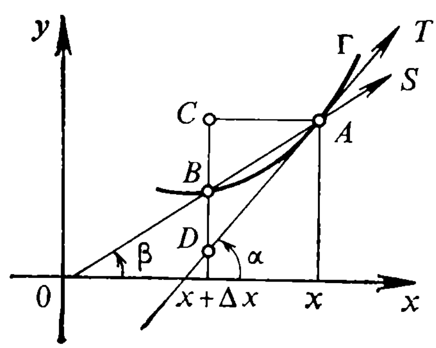

Fig. 38

us mark on Γ another point $B = (x + \Delta x, f(x + \Delta x))$, where $\Delta x \neq 0$ (Fig. 37 shows the case when $\Delta x > 0$, and Fig. 38 when $\Delta x < 0$). The straight line passing through the points A and B in the direction of an increase in x (marked with an arrow) will be called a *secant* and denoted by S. The angle formed by S with the positive direction of the x-axis will be denoted by β. We hold that $-\pi/2 < \beta < \pi/2$. For $\beta > 0$ the angle is measured from the x-axis anticlockwise and for $\beta < 0$ clockwise. In the given figures $\beta > 0$. In Figure 37 $\Delta x = AC$, $\Delta y = CB$, and in Fig. 38 $\Delta x = -AC$, $\Delta y = -CB$. In both cases $\Delta y/\Delta x = \tan \beta$.

If $\Delta x \to 0$, then $\Delta y \to 0$ and the point B, moving along Γ, tends to A. If in this situation the angle β

tends to a certain value α different from $\pi/2$ and $-\pi/2$, then there exists the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\beta \to \alpha} \tan \beta = \tan \alpha, \tag{1}$$

equal to the derivative (finite!) of f at the point x:

$$f'(x) = \tan \alpha. \tag{2}$$

Conversely, if there exists a (finite) derivative $f'(x)$, then $\beta \to \alpha = \operatorname{arc\,tan} f'(x)$.

As β tends to α, the secant S approaches the position occupied by the directed straight line T passing through the point A and making an angle α with the positive direction of the x-axis.

The directed line T is called the *tangent (line) to the curve* Γ *at its point* A.

Definition. *The tangent line to a curve* Γ $(y = f(x))$ *at its point* $A = (x, f(x))$ *is the directed line* T *to which there approaches the secant* S *(a straight line drawn in the direction in which* x *increases) passing through the points* A *and* $B = (x + \Delta x, f(x + \Delta x)) \in \Gamma$, *when* $\Delta x \to 0$.

We have proved that *if a continuous function* $y = f(x)$ *has a finite derivative* $f'(x)$ *at a point* x, *then its graph* Γ *has at its corresponding point a tangent line with the slope* $\tan \alpha = f'(x)$ $(-\pi/2 < \alpha < \pi/2)$.

Conversely, the existence of the limit

$$\lim \beta = \alpha \ (-\pi/2 < \alpha < \pi/2)$$

implies the existence of a finite derivative $f'(x)$ and the validity of equalities (1), (2).

It may happen that f has at x a right and left derivatives different from each other:

$$f'(x^-) \neq f'(x^+).$$

Then A is a *point of break* of Γ *(a corner point)*. In this case the tangent line to Γ at A does not exist, but we may say that there exist a right and left tangents with different slopes:

$$\tan \alpha_1 = \lim_{\substack{\Delta x \to 0 \\ \Delta x < 0}} \frac{\Delta y}{\Delta x} = f'(x^-), \quad \tan \alpha_2 = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{\Delta y}{\Delta x} = f'(x^+).$$

Figure 39 illustrates such a case.

Let now the derivative of f be infinite at x:

$$f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\infty.$$

Note the following four important cases:

$$(1)\quad f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=+\infty,\qquad \beta\to\frac{\pi}{2}\ \text{(Fig. 40)}.$$

$$(2)\quad f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=-\infty,\qquad \beta\to-\frac{\pi}{2}\ \text{(Fig. 41)}.$$

$$(3)\quad f'(x^-)=\lim_{\substack{\Delta x\to 0\\ \Delta x<0}}\frac{\Delta y}{\Delta x}=-\infty,\qquad \beta\to-\frac{\pi}{2}.$$

$$f'(x^+)=\lim_{\substack{\Delta x\to 0\\ \Delta x>0}}\frac{\Delta y}{\Delta x}=+\infty,\qquad \beta\to\frac{\pi}{2}.$$

The left tangent is perpendicular to the x-axis and is directed downward, while the right tangent is perpen-

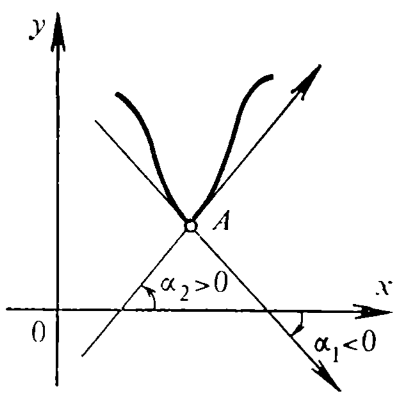

Fig. 39

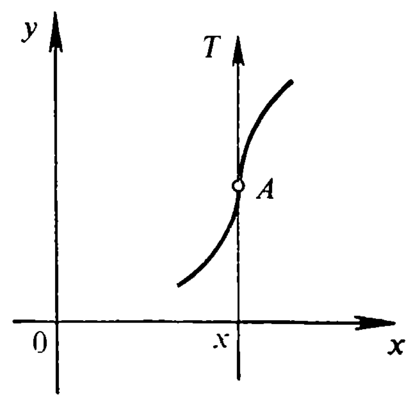

Fig. 40

dicular to the x-axis and is directed upward (Fig. 42).

$$(4)\quad f'(x^-)=\lim_{\substack{\Delta x\to 0\\ \Delta x<0}}\frac{\Delta y}{\Delta x}=+\infty,\qquad \beta\to\frac{\pi}{2},$$

$$f'(x^+)=\lim_{\substack{\Delta x\to 0\\ \Delta x>0}}\frac{\Delta y}{\Delta x}=-\infty,\qquad \beta\to-\frac{\pi}{2}.$$

Here, the left and the right tangents are directed parallel to the y-axis, the former upward and the latter downward (see Fig. 43).

Remark. The ordinary definition of the tangent (line) to a curve is as follows: The tangent line T to a curve Γ at its point A is the limiting position, if this exists, of the secant line S passing through a fixed point A on Γ and a variable point $B \in \Gamma$ as B tends to A along Γ.

It is not assumed in this definition that S and T are directed straight lines. This definition is quite correct

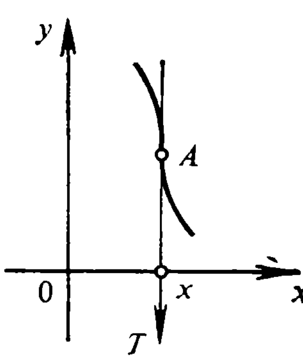

Fig. 41

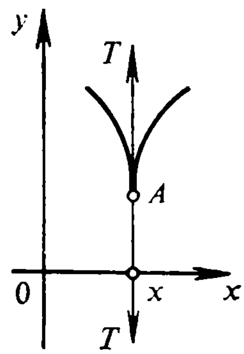

Fig. 42

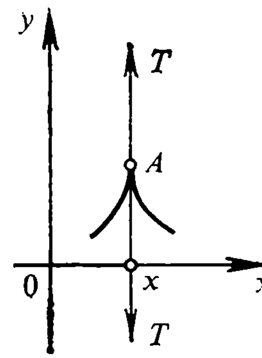

Fig. 43

in the case of a tangent line not parallel to the y-axis. But if it is applied, say, to Case (4) (see Fig. 43, where A is a corner point), then we obtain that the given curve has one tangent line at point A. This contradicts our idea of the smoothness of a curve having a tangent line.

According to the above definition, at point A the curve has two oppositely directed, coincident tangent lines, the angle between them being equal to π.

We know from analytical geometry that the equation of a straight line (in the plane) passing through a point (x_0, y_0) at an angle α to the positive direction of the x-axis $(-\pi/2 < \alpha < \pi/2)$ has the form* $y - y_0 = m(x - x_0)$ $(m = \tan \alpha)$. Hence, the equation of the tangent line to the curve $y = f(x)$ at the point (x_0, y_0) has the form

$$y - y_0 = y_0' (x - x_0), \tag{3}$$

where

$$y_0 = f(x_0), \ y_0' = f'(x_0).$$

* See our book *Higher Mathematics. Fundamentals of Linear Algebra and Analytical Geometry*, Sec. 8.

10-01779

A straight line passing through a point $A \in \Gamma$ perpendicular to the tangent line to Γ at this point is called the *normal to* Γ *at point* A. Its equation, obviously, has the form

$$y - y_0 = -\frac{1}{y_0'}(x - x_0). \tag{4}$$

Example 1*. Find the equation of the tangent line to the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (-a \leqslant x \leqslant a) \tag{5}$$

at its point (x_0, y_0), i.e. $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$.

Curve (5) is called the *ellipse*. It is obvious that the ellipse is symmetric with respect to the coordinate axes, since its equation remains unchanged when x is replaced by $(-x)$ and y by $(-y)$. When deriving the equation of the tangent, we shall assume that $-a \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$. From (5) we have

$$y = \frac{b}{a}\sqrt{a^2 - x^2}. \tag{5'}$$

Hence,

$$y' = \frac{-bx}{a\sqrt{a^2 - x^2}}.$$

Let us evaluate the function y and the derivative y' at the point x_0:

$$y_0 = y(x_0) = \frac{b}{a}\sqrt{a^2 - x_0^2}, \qquad y'(x_0) = \frac{-bx_0}{a\sqrt{a^2 - x_0^2}},$$

$$y_0 y'(x_0) = -\frac{b^2}{a^2}x_0. \tag{6}$$

The equation of the tangent line to the ellipse at the point (x_0, y_0):

$$Y - y_0 = y'(x_0)(X - x_0). \tag{7}$$

* Examples 1, 2, 3 may be considered in Sec. 4.8 after mastering the technique of differentiation.

Multiplying (7) by y_0/b^2, by virtue of (6) we shall have:

$$\frac{Yy_0}{b^2}-\frac{y_0^2}{b^2}=y'(x_0)\left(\frac{Xy_0}{b^2}-\frac{x_0y_0}{b^2}\right),\quad \frac{Yy_0}{b^2}-\frac{y_0^2}{b^2}$$
$$=-\frac{Xx_0}{a^2}+\frac{x_0^2}{a^2}.$$

Since $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$, the equation of the tangent will be written as follows:

$$\frac{Xx_0}{a^2}+\frac{Yy_0}{b^2}=1. \tag{8}$$

Thus, in order to obtain the equation of the tangent to the ellipse at its point (x_0, y_0), we have to substitute Xx_0 for x^2 and Yy_0 for y^2 in the equation of ellipse (5).

For negative values of y $(-b \leqslant y < 0)$ we reason just in the same manner and, hence, (8) will be the equation

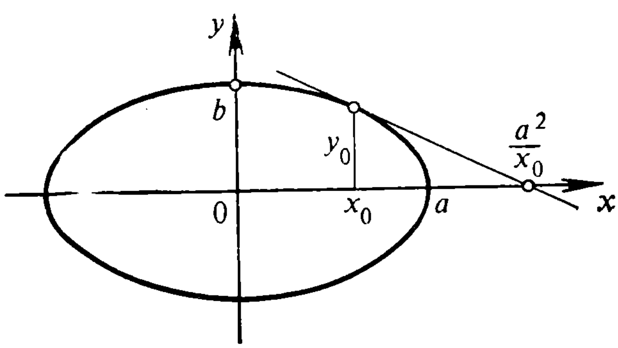

Fig. 44

of the tangent at any point (x_0, y_0) of the ellipse. It is seen from equation (8) that the tangent to the ellipse at its point (x_0, y_0) intersects the x-axis at the point with the abscissa a^2/x_0, that is, for $x_0 > 0$ this point of intersection lies on the right of the ellipse, while for $x_0 < 0$ it is found on its left (Fig. 44).

Example 2. Find the equation of the tangent to the curve

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad (|x| \geqslant a) \tag{9}$$

at its point (x_0, y_0) $\left(\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}=1\right)$.

Curve (9) is called the *hyperbola*. This curve is also symmetric with respect to the coordinate axes.

Reasoning as in Example 1, we get the equation of the tangent to the hyperbola in the form

$$\frac{Xx_0}{a^2} - \frac{Yy_0}{b^2} = 1 \quad (|x_0| \geqslant a).$$

The point at which this tangent intersects the x-axis has the abscissa a^2/x_0 $\left(0 < \frac{a^2}{|x_0|} \leqslant a\right)$, i.e. this point

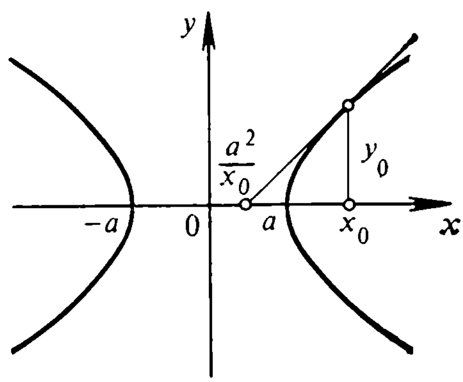

Fig. 45

of intersection lies in $(0, a]$ for $x_0 \geqslant a$ and in $[-a, 0)$ for $x_0 \leqslant -a$ (Fig. 45).

Example 3. Find the equation of the tangent to the curve

$$y^2 = 2px \ (x \geqslant 0, \ p > 0) \tag{10}$$

at its point (x_0, y_0) $(y_0^2 = 2px_0)$.

The given curve is called the *parabola*; it is arranged symmetrically about the x-axis (i.e. in (10) x is an even function of y). Therefore, it is sufficient to consider the upper half of the parabola $(y > 0)$. From (10) we have

$$y = \sqrt{2px}. \tag{10'}$$

Hence,

$$y' = \frac{p}{\sqrt{2px}}, \quad y_0 = \sqrt{2px_0}, \quad y'(x_0) = \frac{p}{\sqrt{2px_0}} = \frac{p}{y_0}.$$

The equation of the tangent line to the parabola at the point (x_0, y_0):

$$Y - y_0 = y'(x_0)(X - x_0)$$

or

$$Y - y_0 = \frac{p}{y_0}(X - x_0), \quad Yy_0 - y_0^2 = pX - px_0.$$

Since $y_0^2 = 2px_0$, we have

$$Yy_0 = p(X + x_0). \tag{11}$$

Thus, in order to get the equation of the tangent line to the parabola at its point (x_0, y_0), we have to replace y^2 by Yy_0 and $2x$ by $X + x_0$ in the equation of parabola (10).

Tangent (11) to parabola (10′) at its point (x_0, y_0) intersects the x-axis at the point with the abscissa $(-x_0)$

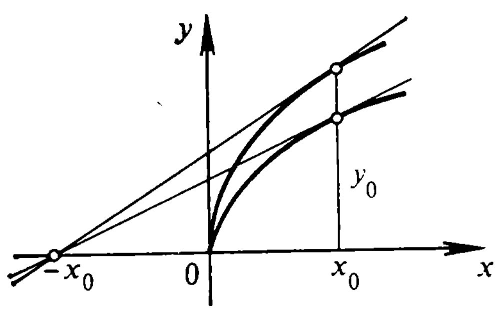

Fig. 46

(Fig. 46) irrespective of the quantity p, i.e. tangent lines to any parabolas $y^2 = 2px$ at the point $(x_0, \sqrt{2px_0})$ intersect the x-axis at one and the same point $(-x_0)$.

Sec. 4.3. Derivatives of Elementary Functions

The constant function C: here to every x there corresponds one and the same value $y = C$. Hence, to the value $x + \Delta x$ there corresponds the value $y + \Delta y = C$. Consequently,

$$C' = \lim_{\Delta x \to 0} \frac{C - C}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0. \tag{1}$$

The power function x^n $(n = 1, 2, \ldots)$.

$$(x^n)' = nx^{n-1}, \tag{2}$$

since

$$\frac{1}{\Delta x}[(x+\Delta x)^n - x^n]$$

$$= \frac{1}{\Delta x}\left[x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}\Delta x^2 + \ldots + \Delta x^n - x^n\right]$$

$$= nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\Delta x + \ldots + \Delta x^{n-1} \xrightarrow[\Delta x \to 0]{} nx^{n-1}.$$

The following formulas are valid:

$$(u \pm v)' = u' \pm v', \tag{3}$$

$$(uv)' = uv' + u'v, \tag{4}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \quad (v \neq 0). \tag{5}$$

It is assumed here that $u = u(x)$, $v = v(x)$ are functions of x having a derivative at a point x. In Case (5) it is additionally assumed that $v(x) \neq 0$. It is stated that in such an event at the point x there exist derivatives standing on the left of equalities (3), (4), (5), and these equalities are true.

Indeed, let there be given Δx. Then to the new value $x + \Delta x$ of the argument there correspond new values of our functions $u + \Delta u$, $v + \Delta v$ and

$$\Delta(u \pm v) = [(u + \Delta u) \pm (v + \Delta v)] - (u \pm v)$$

$$= \Delta u \pm \Delta v,$$

$$(u \pm v)' = \lim_{\Delta x \to 0} \frac{\Delta(u \pm v)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \pm \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = u' \pm v'.$$

Further (explained below),

$$\Delta(uv) = (v + \Delta u)(v + \Delta v) - uv = u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v,$$

$$(uv)' = \lim_{\Delta x \to 0} \frac{\Delta(uv)}{\Delta x} = \lim_{\Delta x \to 0} \frac{u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v}{\Delta x}$$

$$= u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \Delta u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$

$$= uv' + vu' + 0v' = uv' + vu'.$$

It should be taken into consideration that, since the function u has a derivative, it is continuous and, therefore, $\Delta u \to 0$ as $\Delta x \to 0$.

Finally,

$$\left(\frac{u}{v}\right)' = \lim_{\Delta x \to 0} \left(\frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}\right)\frac{1}{\Delta x} = \lim_{\Delta x \to 0} \frac{v\,\Delta u - u\,\Delta v}{(v+\Delta v)\,v\,\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{(v+\Delta v)\,v} = \frac{u'v - uv'}{v^2}.$$

Again, it should be taken into account that $\Delta v \to 0$ as $\Delta x \to 0$, because the function v is continuous, since it has a derivative.

Consider the function $\sin x$. Then

$$(\sin x)' = \cos x, \tag{6}$$

since

$$(\sin x)' = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2\sin\dfrac{\Delta x}{2}\cos\left(x+\dfrac{\Delta x}{2}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin\dfrac{\Delta x}{2}}{\dfrac{\Delta x}{2}} \lim_{\Delta x \to 0} \cos\left(x+\frac{\Delta x}{2}\right) = 1\cdot\cos x = \cos x.$$

It should be borne in mind that the function $\cos x$ is continuous.

Analogously, we prove that

$$(\cos x)' = -\sin x, \tag{7}$$

$$(\tan x)' = \sec^2 x = \frac{1}{\cos^2 x}, \tag{8}$$

$$(\cot x)' = -\csc^2 x = \frac{-1}{\sin^2 x}. \tag{9}$$

Indeed, for instance,

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x\,(\sin x)' - \sin x\,(\cos x)'}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

For the function $y = \log_a x$ $(x > 0)$, we have

$$\frac{\Delta y}{\Delta x} = \frac{\log_a (x+\Delta x) - \log_a x}{\Delta x} = \frac{\log_a \left(1+\frac{\Delta x}{x}\right)}{\Delta x}$$

$$= \frac{1}{x} \frac{\log_a \left(1+\frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}}.$$

Using the remarkable limit

$$\lim_{u \to 0} \frac{\log_a (1+u)}{u} = \log_a e,$$

we get

$$(\log_a x)' = \frac{1}{x} \log_a e = \frac{1}{x \ln a}. \tag{10}$$

In particular,

$$(\ln x)' = \frac{1}{x}. \tag{10'}$$

Sec. 4.4. Derivative of Composite Function

Theorem 1. *If the function $x = \varphi(t)$ has a derivative at the point t and the function $y = f(x)$ has a derivative at the point x, then the composite function*

$$y = F(t) = f[\varphi(t)] \tag{1}$$

has a derivative (with respect to t) at the point t and the following equality is valid:

$$F'(t) = f'(x)\, \varphi'(t) \tag{2}$$

or

$$y'_t = y'_x x'_t. \tag{3}$$

Proof. Let us take t to which there corresponds the value $x = \varphi(t)$. Assigning to t an increment $\Delta t \neq 0$, we shall obtain an increase in x: $\Delta x = \varphi(t + \Delta t) - \varphi(t)$. If $\Delta x \neq 0$, then

$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x} \frac{\Delta x}{\Delta t}. \tag{4}$$

For $\Delta t \to 0$ Δx will tend to zero because the function $x = \varphi(t)$ is continuous at t, since it has a derivative at this point. Therefore

$$y'_t = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = y'_x x'_t \qquad (5)$$

and formula (3) has been proved for the condition that $\Delta x \neq 0$, or, more precisely, for the condition that there exists a $\delta > 0$ such that $0 < |\Delta x|$ for all Δt for which $0 < |\Delta t| < \delta$.

If this condition is not fulfilled, then formula (3) is proved in another way. This is how we may reason in this case. If the indicated condition is not fulfilled, then there exists a sequence $\{\Delta t_k\}$ convergent to zero such that $\Delta x_k = 0$ for all $k = 1, 2, \ldots,$ and then

$$\lim_{\Delta t_k \to 0} \frac{\Delta x_k}{\Delta t_k} = \lim_{\Delta t_k \to 0} \frac{0}{\Delta t_k} = 0. \qquad (6)$$

This shows that $x'_t = 0$, since the limit $\frac{\Delta x_k}{\Delta t_k}$ must be one and the same number irrespective of the way in which Δt_k tends to zero. But it was assumed that the derivative x'_t exists. Let us separate two kinds of sequences $\{\Delta t_k\}$ tending to zero. Grouped under the first kind are such sequences in which $\Delta x_k \neq 0$ for all $k = 1, 2, \ldots$. For them, as in equality (5),

$$\lim_{\Delta t_k \to 0} \frac{\Delta y_k}{\Delta t_k} = \lim_{\Delta x_k \to 0} \frac{\Delta y_k}{\Delta x_k} \lim_{\Delta t_k \to 0} \frac{\Delta x_k}{\Delta t_k} = y'_x x'_t = 0. \qquad (7)$$

Let the sequences $\{\Delta t_k\}$ of the second kind be such that $\Delta x_k = 0$ $(k = 1, 2, \ldots)$. Then

$$\Delta y_k = F(t + \Delta t_k) - F(t) = f(x + \Delta x_k) - f(x)$$
$$= f(x) - f(x) = 0$$
$$(k = 1, 2, \ldots)$$

and

$$\lim_{\Delta t_k \to 0} \frac{\Delta y_k}{\Delta t_k} = \lim_{\Delta t_k \to 0} \frac{0}{\Delta t_k} = 0. \qquad (8)$$

If now an arbitrary sequence $\{\Delta t_k\}$ is given which tends to zero, then we choose out of it those Δt_k for

which $\Delta x_k \neq 0$. If it happens that there are infinitely many such k's, then they form a sequence of the first kind for which (7) was proved. The remaining Δt_k can, in their turn, form a sequence; then this will be a sequence of the second kind for which (8) is proved. But then for an arbitrary sequence $\{\Delta t_k\}$ with $\Delta t_k \to 0$

$$\lim_{\Delta t_k \to 0} \frac{\Delta y_k}{\Delta t_k} = 0 = y'_x \cdot x'_t,$$

and we have proved the theorem completely.

Formula (1) can be complicated. For instance, if $z = f(y)$, $y = \varphi(x)$, $x = \psi(\xi)$ and all three functions have derivatives at the corresponding points, then $z'_\xi = z'_y y'_x x'_\xi$.

Example 1. $y = \ln \sin^2 x$ $(x \neq 0)$.

Making $y = \ln u$, $u = v^2$, $v = \sin x$, we get

$$y'_x = y'_u u'_v v'_x = \frac{1}{u} 2v \cos x = \frac{2 \sin x \cos x}{\sin^2 x} = 2 \cot x.$$

Example 2. $y = \sin(x^2 + 2x - 1)$.

We make $u = x^2 + 2x - 1$. Then

$$y'_x = y'_u \cdot u'_x = \cos u \cdot (2x + 2)$$
$$= 2(x + 1) \cos(x^2 + 2x - 1).$$

When carrying out computations, the auxiliary variables are usually not introduced; they are only implied.

And so, in the case of Example 1 we have:

$$y'_x = \frac{1}{\sin^2 x} (\sin^2 x)' = \frac{1}{\sin^2 x} 2 \sin x (\sin x)'$$
$$= \frac{2}{\sin x} \cos x = 2 \cot x.$$

Or even briefer

$$y'_x = \frac{1}{\sin^2 x} 2 \sin x \cos x = 2 \cot x.$$

Sec. 4.5. Derivative of Inverse Function

Let a function $y = f(x)$ be strictly increasing, continuous on an interval (a, b), and have a finite nonzero derivative $f'(x)$ at a certain point $x \in (a, b)$; then its inverse $x = f^{-1}(y) = g(y)$ also has a derivative at the corresponding point

which is defined by the equality

$$g'(y) = \frac{1}{f'(x)} \tag{1}$$

or

$$x'_y = \frac{1}{y'_x}. \tag{1'}$$

Proof. As we know, the inverse function $x = g(y)$ is strictly increasing and continuous on (A, B), where

$$A = \inf_{x \in (a, b)} f(x), \quad B = \sup_{x \in (a, b)} f(x)$$

(see Sec. 3.6, Theorem 1').

Let us give y under consideration an increment $\Delta y \neq 0$. Then the inverse function receives the corresponding increment Δx which is also different from zero due to strict monotonicity of f. Therefore

$$\frac{\Delta x}{\Delta y} = \frac{1}{\dfrac{\Delta y}{\Delta x}}.$$

If now $\Delta y \to 0$, then the increment Δx also tends to zero, since $g(y)$ is continuous; but as $\Delta x \to 0$ $\frac{\Delta y}{\Delta x} \to f'(x) \neq 0$, consequently there exists the limit

$$\lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = \frac{1}{\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}} = \frac{1}{f'(x)}.$$

This proves formula (1).

Remark. *If $f'(x) \neq 0$ is continuous on (a, b), then $g'(y)$ is continuous on (A, B).*

This follows from (1), where we may set $x = g(y)$:

$$g'(y) = \frac{1}{f'[g(y)]}, \quad y \in (A, B).$$

In fact, the composite function $f'[g(y)]$, consisting of continuous functions f' and g, is continuous.

Sec. 4.6. Derivatives of Elementary Functions, Continued

1. $y = a^x$. Hence $x = \log_a y$ is an inverse function. Therefore

$$y'_x = \frac{1}{x'_y} = \frac{1}{\frac{1}{y \ln a}} = y \ln a = a^x \ln a, \text{ i.e. } (a^x)' = a^x \ln a.$$

In particular,

$$(e^x)' = e^x, \quad (e^{-x})' = -e^{-x}.$$

2. $y = \arcsin x$ ($|x| < 1$, $-\pi/2 < y < \pi/2$), $x = \sin y$ is its inverse. Therefore

$$y'_x = \frac{1}{x'_y} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}},$$

i.e.

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}.$$

The radical is taken with the plus sign, since $\cos y > 0$ on $(-\pi/2, \pi/2)$.

3.

$$(\arccos x)' = \left(\frac{\pi}{2} - \arcsin x\right)' = -\frac{1}{\sqrt{1-x^2}}.$$

4. $y = \arctan x$, its inverse being $x = \tan y$ ($-\infty < x < \infty$, $-\pi/2 < y < \pi/2$). Then

$$(\arctan x)' = \frac{1}{(\tan y)'} = \cos^2 y = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2},$$

i.e.

$$(\arctan x)' = \frac{1}{1+x^2}.$$

5. Similarly, we prove that

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}.$$

6. The *derivative of the power function* x^α ($x > 0$, α is an arbitrary real number). We have

$$x^\alpha = e^{\alpha \ln x}.$$

Since the functions e^u and $\alpha \ln x$ have a derivative, by the theorem on the derivative of a composite function, we get

$$(x^\alpha)' = (e^{\alpha \ln x})' = e^{\alpha \ln x} \cdot \frac{\alpha}{x} = \alpha \frac{x^\alpha}{x} = \alpha x^{\alpha-1}.$$

Hence,

$$(x^\alpha)' = \alpha x^{\alpha-1}.$$

This result agrees with formula (2) from Sec. 4.3 for the derivative of the function x^n $(x \in (-\infty, \infty))$, where n is a natural number.

7. The *function* $y = u(x)^{v(x)}$ $(u > 0)$. If $u(x)$ and $v(x)$ have a derivative, then the function

$$u^v = e^{v \ln u} \tag{1}$$

also has a derivative and

$$(u^v)' = e^{v \ln u}(v \ln u)' = u^v \left(\frac{v}{u} u' + v' \ln u\right). \tag{2}$$

The expression

$$[\ln f(x)]' = \frac{f'(x)}{f(x)} \tag{3}$$

is called the *logarithmic derivative of the function* f.

Since

$$\ln u^v = v \ln u,$$

by formula (3), we have

$$\frac{(u^v)'}{u^v} = (v \ln u)' = v' \ln u + \frac{vu'}{u},$$

whence (2) follows.

8. *Hyperbolic functions.*

$$(\sinh x)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh x,$$

$$(\cosh x)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{e^x - e^{-x}}{2} = \sinh x,$$

$$(\tanh x)' = \left(\frac{\sinh x}{\cosh x}\right)' = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x},$$

$$(\coth x)' = \left(\frac{\cosh x}{\sinh x}\right)' = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x}(x \neq 0).$$

9. $y = \operatorname{Arsinh} x$ is the inverse of the function $x = \sinh y$. Hence

$$(\operatorname{Arsinh} x)' = \frac{1}{(\sinh y)'} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

(see Example 2 in Sec. 4.12).

Sec. 4.7. Differential of a Function

Let a function f have a finite derivative at a point x

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x).$$

Then, for sufficiently small Δx, $\frac{\Delta y}{\Delta x}$ can be written in the form of a sum of $f'(x)$ and a certain function which will be denoted by $\varepsilon(\Delta x)$. The latter function possesses the following property: it tends to zero together with Δx

$$\frac{\Delta y}{\Delta x} = f'(x) + \varepsilon(\Delta x) \quad (\varepsilon(\Delta x) \to 0, \quad \Delta x \to 0)$$

and the increment of f at the point x can be written in the form

$$\Delta y = f'(x)\,\Delta x + \Delta x \cdot \varepsilon(\Delta x) \quad (\varepsilon(\Delta x) \to 0,\ \Delta x \to 0)$$

or

$$\Delta y = f'(x)\,\Delta x + \underset{\Delta x \to 0}{o(\Delta x)}. \tag{1}$$

It should be borne in mind that the expression $\underset{\Delta x \to 0}{o(\Delta x)}$ is understood as a function of Δx such that its ratio to Δx tends to zero together with Δx.

Definition. *A function f is said to be differentiable at a point x if its increment Δy at that point can be represented in the form*

$$\Delta y = A \cdot \Delta x + \underset{\Delta x \to 0}{o(\Delta x)}, \tag{2}$$

where A is a constant independent of Δx, but, generally, dependent on x.

Theorem 1. *For a function f to be differentiable at a point x, i.e. in order for its increment at this point to be represent-*

able by formula (2), *it is necessary and sufficient that it have a finite derivative at this point. And then* $A = f'(x)$.

Hence, to say that f has a derivative at a point x means the same as "f is differentiable at the point x". Therefore, the process of finding a derivative is also called the *differentiation of a function.*

The Proof of Theorem 1. The sufficiency of the condition is proved above: from the fact of existence of a finite derivative $f'(x)$ there followed the possibility of representing Δy in form (1), where we may set $f'(x) = A$.

The necessity of the condition. Let the function f be differentiable at the point x. Then from (2), assuming that $\Delta x \neq 0$, we obtain

$$\frac{\Delta y}{\Delta x} = A + \frac{\underset{\Delta x \to 0}{o(\Delta x)}}{\Delta x} = A + \underset{\Delta x \to 0}{o(1)}.$$

As $\Delta x \to 0$ the limit of the right-hand member exists and equals A:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = A.$$

This means that there exists the derivative

$$f'(x) = A.$$

Let the function $y = f(x)$ be differentiable at the point x, i.e. for its increment Δy at this point equality (2) is fulfilled. Then Δy is the sum of two terms. The first of them $A\,\Delta x$ is proportional to Δx: in such cases it is said to be a linear homogeneous function of Δx. The second term $\underset{\Delta x \to 0}{o(\Delta x)}$ is an infinitely small function of a higher order of smallness as compared with Δx. If $A \neq 0$, then the second term tends to zero faster than the first term as $\Delta x \to 0$. That is why the first term $A\,\Delta x = f'(x)\,\Delta x$ is called the *principal linear part of the increment* Δy (as $\Delta x \to 0$!. See the definition at the end of Sec. 3.10). This term is also called the *differential of a function* and is denoted by dy. Thus, by definition,

$$dy = df = f'(x)\,\Delta x.$$

Figure 47 shows the graph Γ of the function $y = f(x)$; T is the tangent line to Γ at the point A having the abscissa x; $f'(x) = \tan \alpha$, where α is the angle made by the tangent line with the x-axis;

$$dy = f'(x)\,\Delta x = \tan \alpha\, \Delta x = CD, \quad DB = \Delta y - dy = \underset{\Delta x \to 0}{o(\Delta x)}.$$

Hence, *the differential of a function y at a point x corresponding to the increment Δx is an increment of the ordinate of the point lying on the tangent line* ($dy = CD$).

Generally speaking, $dy \neq \Delta y$, since $\Delta y = dy + \underset{\Delta x \to 0}{o(\Delta x)}$, and the second term of this sum is, generally speaking, not equal to zero. The equality $\Delta y = A\,\Delta x = dy$ is valid for any x only in the case of a linear function $y = Ax + B$, for $y = x$, $dy = dx = \Delta x$, i.e. *the differential and the increment of the independent variable x are equal to each other* ($dx = \Delta x$). Therefore, the differential of an arbitrary function f is usually written in the following way:

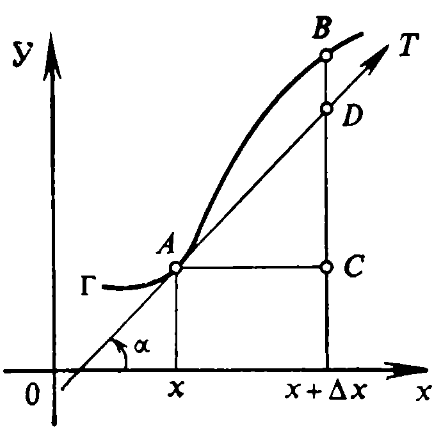

Fig. 47

$$dy = f'(x)\,dx,$$

whence

$$f'(x) = \frac{dy}{dx},$$

i.e. *the derivative of the function f at a point x is equal to the ratio of the differential of the function at this point to the differential of the independent variable x.*

This explains the fact that the expression dy/dx (read "dy over dx") is used as the symbol for denoting the derivative.

It should be borne in mind that the *differential dx of the independent variable is independent of x and is equal to an arbitrary increment Δx of the argument x. As to the differ-*

ential dy of the function y (different from x), it depends both on x and dx.

Note the following formulas:

$$d(u \pm v) = du \pm dv, \tag{3}$$

$$d(u \cdot v) = u\,dv + v\,du, \tag{4}$$

$$d(cu) = c\,du \ (c \text{ constant}), \tag{5}$$

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2} \ (v \neq 0), \tag{6}$$

where it is assumed that u and v are differentiable functions at a point x under consideration.

For instance, formula (6) is proved as follows:

$$d\left(\frac{u}{v}\right) = \left(\frac{u}{v}\right)' dx = \frac{vu'\,dx - uv'\,dx}{v^2} = \frac{v\,du - u\,dv}{v^2}.$$

If a function $y = f(x)$ is differentiable at a point x, then, by formula (1), its increment corresponding to the increment Δx, can be written in the following way:

$$\Delta y = dy + \underset{\Delta x \to 0}{o}(\Delta x).$$

Hence it follows that the differential of a function may serve as a good approximation to the increment of the function for sufficiently small Δx. In this sense we write the approximate equality

$$\Delta y \approx dy = f'(x)\,dx, \tag{7}$$

which is widely used.

Example 1. If we assume that

$$\sqrt[3]{8.001} \approx \sqrt[3]{8} = 2,$$

then the error is approximately equal to the differential of the function $y = x^{1/3}$ at the point $x = 8$ corresponding to the increment $\Delta x = 0.001$:

$$dy = \frac{1}{3} x^{-2/3} \Delta x = \frac{1}{3} 8^{-2/3} \cdot 0.001 = 1/12\,000.$$

The problem concerning the accuracy of our reasoning can be solved by the methods which we shall study later on (see Sec. 4.14).

Sec. 4.8. Another Definition of Tangent

If the derivative $f'(x_0)$ is finite, then we can give another (equivalent) definition of the tangent line.

Let there be given an arbitrary straight line L: $y - y_0 = m(x - x_0)$ passing through the point $A = (x_0, f(x_0))$ belonging

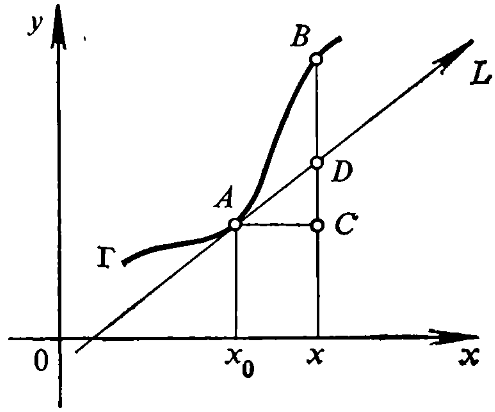

Fig. 48

to the curve Γ: $y = f(x)$; and let $B = (x, f(x))$ be another point on the curve Γ. The distance from B to L in the direction of the y-axis is equal to

$$\rho(x) = |f(x) - y_0 - m(x - x_0)|. \tag{1}$$

(In Fig. 48: $\rho(x) = BD$.)

L is called the tangent line to Γ at the point A if

$$\rho(x) \underset{x \to x_0}{=} o(x - x_0). \tag{2}$$

If the line L is a tangent to Γ at the point A in the sense of the first definition, then $m = f'(x_0)$. Since f is differentiable, we have

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + o(x - x_0), \quad x \to x_0,$$

whence

$$\rho(x) = |f(x) - f(x_0) - f'(x_0)(x - x_0)| = o(x - x_0),$$
$$x \to x_0,$$

i.e. L is a tangent line in the sense of the second definition.

Conversely, let L be a tangent line in the sense of the second definition. Then (see (1) and (2))

$$\rho(x) = |f(x) - f(x_0) - m(x - x_0)| = o(x - x_0), \quad x \to x_0,$$

or, which is the same

$$f(x) - f(x_0) = m(x - x_0) + o(x - x_0) \text{ as } x \to x_0.$$

This shows that the function f is differentiable at the point x_0 and $m = f'(x_0)$. But then L is a tangent line in the sense of the first definition and its equation has the form

$$y - y_0 = f'(x_0)(x - x_0).$$

Remark. It follows from the aforegoing that the curve $y = f(x)$ has a tangent at the point $(x_0, f(x_0))$ if, and only if, the function f is differentiable at the point x_0.

Sec. 4.9. Derivatives of Higher Orders

Let there be given a function f on an open interval (a, b). Its derivative, if it exists on the interval (a, b), is a certain function $f'(x)$. We shall also call it the *first derivative*. But it may happen that the first derivative, in its turn, has a derivative on the interval (a, b). The latter is then called the *second derivative of f* or the *derivative of f of the second order* and is denoted in one of the following ways:

$$f''(x) = f^{(2)}(x) = (f'(x))' \text{ or } y'' = (y')'.$$

In general, the *derivative of a function f of order n* (*or, simply, the nth derivative*) *is defined as the* (*first*) *derivative of the derivative of f of order $n - 1$* and is denoted as

$$f^{(n)}(x) = (f^{(n-1)}(x))' \text{ or } y^{(n)} = (y^{(n-1)})'.$$

Of course, the nth derivative of a given function f at a given point x may or may not exist.

If a function f is said to have the nth derivative at a point x, then it is meant that it possesses the derivative $f^{(n-1)}(x)$ of order $n - 1$ in at least a sufficiently small neighbourhood of the point x and that this derivative itself has the derivative at the point x. It is the latter derivative that is denoted by the symbol $f^{(n)}(x)$ and is called the derivative of the nth order, or the nth derivative of f at the point x.

Examples.

(1) $(e^x)^{(n)} = e^x$.

(2) $(a^x)' = a^x \ln a$, $(a^x)'' = a^x \ln^2 a$, . . ., $(a^x)^{(n)} = a^x \ln^n a$.

(3) $(x^m)' = mx^{m-1}$, $(x^m)'' = m(m-1)x^{m-2}$, . . ., . . ., $(x^m)^{(n)} = m(m-1)\ldots(m-n+1)x^{m-n}$.

If m is a natural number, then, obviously,

$$(x^m)^{(m)} = m! \text{ and } (x^m)^{(n)} = 0 \ (n > m).$$

(4) $$(\sin x)' = \cos x = \sin\left(x + \frac{\pi}{2}\right),$$

$$(\sin x)'' = \left[\sin\left(x + \frac{\pi}{2}\right)\right]' = \sin\left(x + 2\frac{\pi}{2}\right),$$

. . . .

$$(\sin x)^{(n)} = \sin\left(x + n\frac{\pi}{2}\right).$$

(5) $$(\cos x)^{(n)} = \cos\left(x + n\frac{\pi}{2}\right).$$

But, by far not for any function, we succeed in finding a general formula for its nth derivative.

Exercise. Using the method of mathematical induction, prove the Leibniz formula for the derivative of the nth order of the product of two functions:

$$(uv)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(n-k)} v^{(k)},$$

where u and v have derivatives up to order n (inclusive),

$$C_n^k = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!}, \quad 0! = 1.$$

Sec. 4.10. Differentials of Higher Orders. The Invariant Property of the First-order Differential

If a function $y = f(x)$ is defined on an open interval (a, b), then it can be, obviously, represented as a composite function in infinitely many ways:

$$y = \varphi(z), \quad z = \psi(x).$$

Thus, y can be regarded as a function of x ($y = f(x)$) and as a function of z ($y = \varphi(z)$), where z is, in its turn, a function of x ($z = \psi(x)$).

The argument x will be called *independent* to mean that during our reasoning x will not be regarded as a function of some variable, while the argument z will be called *dependent* (on x!).

As we know, the differential of a function $y = f(x)$ at a point x is the product of the derivative of f at this point by the differential of the independent variable:

$$dy = f'(x)\,dx.$$

Here, dx is an arbitrary number *independent of* x. It manifests itself in that the derivative of dx with respect to x is equal to zero:

$$(dx)' = 0.$$

The differential of a function is also called the *first differential of this function.*

By definition, the *second differential of a function* $y = f(x)$ *at the point* x *is the differential of the first differential at this point* and is denoted as follows:

$$d^2y = d\,(dy).$$

In order to compute the second differential, we have to take the derivative of the product $f'(x)\,dx = dy$ with respect to x, regarding dx as a constant (independent of x!), and to multiply the obtained result by dx:

$$d^2y = d\,[f'(x)\,dx] = dx\,d\,[f'(x)] = f''(x)\,dx^2.$$

In general, the *differential of order* n *of the function* $y = f(x)$ *is defined as the first differential of the differential of order* $n - 1$ *of this function*:

$$d^ny = d\,(d^{n-1}y).$$

Obviously,

$$d^ny = f^{(n)}(x)\,dx^n, \tag{1}$$

since this formula is true for $n = 1$, and if we assume that it is true for $n - 1$, then

$$d^ny = d\,[f^{(n-1)}(x)\,dx^{n-1}] = dx^{n-1}d\,[f^{(n-1)}(x)] = f^{(n)}(x)\,dx^n.$$

Of course, for the differential of order n of a function $y = f(x)$ to exist at the point x, it is necessary that this function have a derivative $f^{(n)}(x)$ of order n at this point.

By virtue of (1), we have

$$y_x^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}, \tag{2}$$

i.e. the *nth derivative of the function y with respect to the independent variable x is equal to the ratio of the nth differential of y to $dx^n = (dx)^n$.*

We shall learn later on that formula (2) turns out to be incorrect if the independent variable x is replaced in it by the dependent variable z (see (4) below).

We have defined the differentials of the first and, generally speaking, of a higher order of the function $y = f(x)$, where x is an *independent variable.* But, as it was shown above, the function y can also be written in the form

$$y = \varphi(z),$$

where z is a certain function of x ($z = \psi(x)$, $f(x) = \varphi[\psi(x)]$). There arises a question how the introduced differentials are expressed in terms of the dependent variable z.

For the first differential this question is answered in the following way:

$$dy = y'_x\,dx = y'_z z'_x\,dx = y'_z\,(z'_x\,dx) = y'_z\,dz.$$

We see that the differential of the function y is equal to the product of its derivative y'_z by dz:

$$dy = y'_z\,dz, \tag{3}$$

i.e. the *first differential of the function y is expressed by one and the same formula irrespective of whether y is regarded as a function of the independent variable x or of the dependent variable z.*

The form of the first differential (see (3)) is retained, therefore, the *first differential is said to have an invariant form* or that *it has the invariant property.*

But it is not so with the differential of a higher order. Indeed, regarding y as a function of z ($y = \varphi(z)$), we get (see (6) in Sec. 4.7)

$$d^2y = d\,(dy) = d\,[\varphi'(z)\,dz] = dz\,d\,(\varphi'(z)) + \varphi'(z)\,d\,(dz)$$
$$= \varphi''(z)\,dz^2 + \varphi'(z)\,d^2z. \tag{4}$$

In the last equality we have used the invariant property of the first differential, by virtue of which $d(\varphi'(z)) = \varphi''(z)\,dz$. Besides, we have taken into account that $d(dz) = d^2z$. Generally speaking, the quantity d^2z should not be neglected, since it is determined by the equality $d^2z = \psi''(x)\,dx^2$. Its right-hand member is equal to zero (for all x's) only if $\psi(x)$ is a linear function ($\psi(x) = Ax + B$).

We see that (expressed in terms of z) the form of the second differential has not retained: the term $\varphi'(z)\,d^2z$, which is, generally speaking, not equal to zero has been added to the number $\varphi''(z)\,dz^2$.

Sec. 4.11. Differentiation of Functions Represented Parametrically

Let the dependence of y on x be expressed by the parameter t:

$$\left.\begin{matrix} x = \varphi(t) \\ y = \psi(t) \end{matrix}\right\}, \quad t \in (a, b). \tag{1}$$

This should be understood in the sense that there exists an inverse of the function $x = \varphi(t)$ and we can write the explicit form of dependence of y on x:

$$y = \psi[\varphi^{-1}(x)]. \tag{2}$$

We shall seek for the derivative of y with respect to x in terms of the derivatives of x and y with respect to t. The following notation will be used: y'_x, y''_x, ..., x'_t, y'_t, where the subscript indicates the variable with respect to which the derivative is taken. By virtue of the invariance of the form of the first-order differential, $y'_x = dy/dx$. But $dy = y'_t\,dt$, $dx = x'_t\,dt$. Therefore

$$y'_x = \frac{y'_t}{x'_t} \quad (x'_t \neq 0). \tag{3}$$

For the derivative of the second order we obtain

$$y''_x = \frac{d}{dx}\,y'_x = \frac{d}{dx}\left(\frac{y'_t}{x'_t}\right) = \frac{d}{dt}\left(\frac{y'_t}{x'_t}\right)\frac{dt}{dx} = \frac{x'_t y''_t - y'_t x''_t}{(x'_t)^3}. \tag{4}$$

Proceeding in a similar way, we can obtain formulas for the derivatives $y_x^{(n)}$ with respect to x of order $n > 2$ in terms of the derivatives of x and y with respect to t.

Sec. 4.12. Mean-value Theorems for Derivatives

By definition, a *function f attains a local maximum (minimum) at a point $x = c$ if there exists a neighbourhood of this point $U(c) = (c - \delta, c + \delta)$, in which the following inequality is fulfilled:*

$$f(c) \geqslant f(x), \quad \forall x \in U(c) \tag{1}$$

$$(\text{respectively } f(x) \geqslant f(c), \ \forall x \in U(c)). \tag{1'}$$

A local maximum or a local minimum is called a *local extremum.*

Remark 1. If a function f is continuous on a closed interval $[a, b]$ and attains a maximum (a minimum) on this interval at a point $c \in (a, b)$, then c is, at the same time the point of a local maximum (minimum) of f. It is quite a different thing when a maximum (minimum) of f is attained on $[a, b]$ at one of its end points. Such point is not a point of a local maximum (minimum) of f, since f is not defined in its complete neighbourhood (from the right and from the left at it).

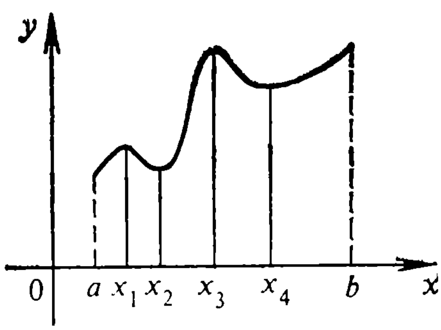

Fig. 49

Figure 49 represents the graph of a function $y = f(x)$ continuous on $[a, b]$. The points x_2 and x_4 are points of a local minimum of f, and x_1, x_3 are points of a local maximum of f. Of course, we may say that b is a point of a local one-sided maximum of f, while a is a local one-sided minimum of f. But a is not a point of local minimum and b is not a point of local maximum.

Theorem 1 (Fermat's Theorem)*. *If a function f has*

* Fermat, Pierre de (1601-1665). French mathematician.

a derivative at a point c and attains a local extremum at this point, then $f'(c) = 0$.

Proof. For the sake of definiteness, we shall assume that f has a local maximum at the point c. By the definition of the derivative, we have

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}.$$

Since $f(c) \geqslant f(x)$, $\forall x \in U(c)$, for sufficiently small $\Delta x > 0$, we have

$$\frac{f(c+\Delta x) - f(c)}{\Delta x} \leqslant 0,$$

whence in the limit, as $\Delta x \to 0$, we obtain that

$$f'(c) \leqslant 0. \tag{2}$$

And if $\Delta x < 0$, then

$$\frac{f(c+\Delta x) - f(c)}{\Delta x} \geqslant 0,$$

therefore, passing to the limit in this inequality as $\Delta x \to 0$ we obtain that

$$f'(c) \geqslant 0. \tag{3}$$

It follows from (2) and (3) that $f'(c) = 0$.

Theorem 2 (Rolle's Theorem)*. *If a function* $y = f(x)$ *is continuous on a closed interval* $[a, b]$, *differentiable on* (a, b) *and* $f(a) = f(b)$, *then there exists a point* $\xi \in (a, b)$, *such that* $f'(\xi) = 0$.

Proof. If f is constant on $[a, b]$, then the derivative $f'(\xi) = 0$ for all $\xi \in (a, b)$.

Let us now hold that f is not constant on $[a, b]$. Since f is continuous on $[a, b]$, there exists a point $x_1 \in [a, b]$, at which f attains a maximum on $[a, b]$ (see Sec. 3.5, Theorem 2) and there exists a point $x_2 \in [a, b]$, at which f attains a minimum on $[a, b]$. Both of them cannot be the end points of the closed interval $[a, b]$, since otherwise

$$\max_{x \in [a, b]} f(x) = \min_{x \in [a, b]} f(x) = f(a) = f(b)$$

* Rolle, Michel (1652-1719). French mathematician who was the first to prove this theorem for polynomials.

and f would be constant on $[a, b]$. Consequently, one of the points x_1, x_2 belongs to the interval (a, b). Let us denote it by ξ. At this point a local extremum is attained. Besides, $f'(\xi)$ exists, since $f'(x)$ exists for all $x \in (a, b)$. Therefore, by Fermat's theorem, $f'(\xi) = 0$.

Remark 2. Rolle's theorem also holds for an open interval (a, b), provided that the following relationship is fulfilled:

$$\lim_{\substack{x \to a \\ x > a}} f(x) = \lim_{\substack{x \to b \\ x < b}} f(x).$$

Remark 3. Rolle's theorem does not remain valid if $f'(x)$ does not exist at least at one point of (a, b). Example: $y = |x|$ on $[-1, 1]$. Nor we are allowed to replace the continuity on $[a, b]$ by the continuity on (a, b). This is exemplified by the function

$$y = \begin{cases} 1, & x = 0, \\ x, & 0 < x \leqslant 1. \end{cases}$$

The point $x = 0$ is a point of discontinuity.

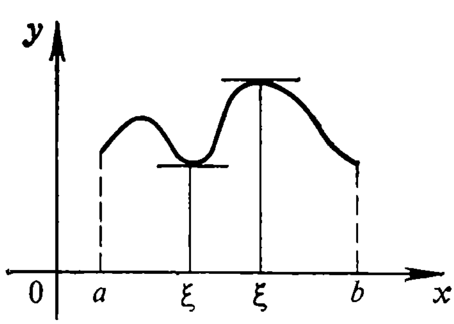

Fig. 50

Remark 4. Rolle's theorem has a simple geometrical meaning. If the conditions of the theorem are fulfilled, then on the graph of the function $y = f(x)$ (Fig. 50) there exists a point $(\xi, f(\xi))$ the tangent line at which is parallel to the x-axis.

Theorem 3 (Cauchy's Theorem). *If two functions $f(x)$ and $g(x)$ are continuous on a closed interval $[a, b]$ and are differentiable on (a, b)* and *$g'(x) \neq 0$ within (a, b), then there exists a point $\xi \in (a, b)$ such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}. \tag{4}$$

Proof. Note that $g(b) - g(a) \neq 0$, since otherwise, by Rolle's theorem, there would exist a point ξ such that $g'(\xi) = 0$ which is impossible. Let us set up an auxiliary

function:

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} [g(x) - g(a)].$$

By virtue of the condition of the theorem, the function F is continuous on $[a, b]$, differentiable on (a, b) and $F(a) = 0$, $F(b) = 0$. Applying Rolle's theorem, we shall obtain that there exists a point $\xi \in (a, b)$ at which $F'(\xi) = 0$. But

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(x),$$

therefore, substituting the point ξ for x, we get the statement of the theorem.

Remark 5. As is clear, it is not necessarily to assume $a < b$ in Cauchy's formula (4). But then $[a, b]$ and (a, b) designate the sets of points x for which $b \leqslant x \leqslant a$, $b < x < a$, respectively.

As a consequence from Cauchy's theorem, for $g(x) = x$, we get Lagrange's theorem.

Theorem 4 (Lagrange's Mean-value Theorem)*. *Let a function $f(x)$ be continuous on a closed interval $[a, b]$ and have the derivative on the open interval (a, b). Then there exists a point c belonging to the interval (a, b) for which the equality*

$$f(b) - f(a) = (b - a) f'(c) \quad (a < c < b) \tag{5}$$

is fulfilled.

Lagrange's theorem has a simple geometrical interpretation if it is written in the form

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad (a < c < b).$$

The left-hand member of this equality is the tangent of the angle of inclination of the chord, subtending the arc of the graph of the function $y = f(x)$ with end points $(a, f(a))$ and $(b, f(b))$, to the x-axis, and the right-hand member is equal to the tangent of the angle of inclination of the tangent line to the graph drawn through the point

* Lagrange, Joseph Louis (1736-1813). Famous French mathematician.

with an intermediate abscissa $c \in (a, b)$. Lagrange's theorem states that if a curve (see Fig. 51) is the graph of a continuous function defined on $[a, b]$ and having the derivative on (a, b), then there is a point on the curve corresponding to a certain abscissa c $(a < c < b)$ such that the tangent line to the curve at this point is parallel to the chord joining the end points $(a, f(a))$ and $(b, f(b))$ of the curve.

Equality (5) is called *Lagrange's formula of finite increments*. The intermediate value c can be conveniently written as

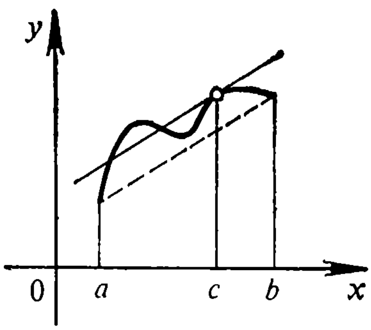

Fig. 51

$$c = a + \theta (b - a),$$

where θ is a number satisfying the inequalities $0 < \theta < 1$. Then Lagrange's formula will take the form

$$f(b) - f(a) = (b - a) f'(a + \theta(b - a)) \quad (0 < \theta < 1). \tag{6}$$

It obviously holds not only for $a < b$, but also for $a \geqslant b$.

Theorem 5. *A function continuous on a closed interval* $[a, b]$, *where* $a < b$, *and having a nonnegative (positive) derivative on the open interval* (a, b) *is nondecreasing (strictly increasing) on* $[a, b]$.

Indeed, let $a \leqslant x_1 < x_2 \leqslant b$; then the conditions of Lagrange's theorem are fulfilled for the closed interval $[x_1, x_2]$ and therefore there is a point c belonging to the open interval (x_1, x_2) for which

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(c) \quad (x_1 < c < x_2).$$

If, by hypothesis, $f' \geqslant 0$ on (a, b), then $f'(c) \geqslant 0$ and

$$f(x_2) - f(x_1) \geqslant 0; \tag{7}$$

if $f' > 0$ on (a, b), then $f'(c) > 0$ and

$$f(x_2) - f(x_1) > 0. \tag{8}$$

Since inequalities (7) and (8) hold for any x_1, x_2, where $a \leqslant x_1 < x_2 \leqslant b$, the function f is nondecreasing on the

interval $[a, b]$ in the first case and strictly increasing on the interval $[a, b]$ in the second case.

Example 1. Let us return to Example 1 from Sec. 4.7, where we had to estimate the quantity $\lambda = \sqrt[3]{8.001} - \sqrt[3]{8}$. Applying Lagrange's formula to the function $\psi(x) = x^{1/3}$, we have

$$\lambda = \psi(8.001) - \psi(8) \doteq 0.001 \cdot \psi'(c) = 0.001 \cdot \frac{1}{3} x^{-2/3}|_{x=c}$$

$$= \frac{1}{3000} c^{-2/3} < \frac{1}{3000} 8^{-2/3} = \frac{1}{12\,000}.$$

In Example 1 of Sec. 4.7 we obtained the same result, but this time it has received a complete proof.

Example 2. The function $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$ has a continuous derivative

$$(\sinh x)' = \frac{1}{2}(e^x + e^{-x}) = \cosh x > 0, \quad \forall x \in (-\infty, \infty),$$

and possesses the following properties:

$$\lim_{x \to -\infty} \sinh x = -\infty, \quad \lim_{x \to +\infty} \sinh x = +\infty.$$

Consequently, it is strictly increasing and continuously differentiable on $(-\infty, \infty)$, and maps the interval $(-\infty, \infty)$ onto $(-\infty, \infty)$. Therefore it has an inverse which is a one-valued, continuously differentiable function denoted as $x = \operatorname{Arsinh} y$, $y \in (-\infty, \infty)$.

Theorem 6. *If a function has a derivative equal to zero on an open interval (a, b), then it is constant on (a, b).*

Indeed, on the basis of Lagrange's theorem, the following equality is valid:

$$f(x) - f(x_1) = (x - x_1) f'(c),$$

where x_1 is a fixed point on the interval (a, b), x is its arbitrary point (which may lie either on the right, or on the left of x_1), and c is a certain point, dependent on x_1 and x, lying between x_1 and x. Since $f'(x) \equiv 0$ on (a, b), we have $f'(c) = 0$ and $f(x) = f(x_1) = C$ for all $x \in (a, b)$.

Note that the conditions imposed on the functions in these theorems cannot be weakened because this may lead to the failure of the assertions of the theorems (see Remarks 1 and 2 to Rolle's theorem).

Definition. *A function $y = f(x)$ is said to be increasing (decreasing) at a point x_0 if there exists a number $\delta > 0$ such that*

$$\frac{\Delta y}{\Delta x} > 0 \left(\frac{\Delta y}{\Delta x} < 0\right) \text{ for } 0 < |\Delta x| < \delta.$$

Theorem 7. *If $f'(x_0) > 0$ (<0), then the function $y = f(x)$ increases (decreases) at the point x_0.*

Proof. Since $f'(x_0) = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, making $\varepsilon > 0$, we can find $\delta > 0$ such that $f'(x_0) - \varepsilon < \frac{\Delta y}{\Delta x} < f'(x_0) + \varepsilon$ for $|\Delta x| < \delta$. Let $f'(x_0) > 0$. Taking $\varepsilon < f'(x_0)$, we obtain that $\frac{\Delta y}{\Delta x} > 0$ for $|\Delta x| < \delta$, i.e. the function f increases at the point x_0.

Remark 6. If a function f has a derivative and is not decreasing on (a, b), then $f'(x) \geqslant 0$ on this interval. Under the mentioned conditions it is impossible for the derivative of f to be negative, at some point $x \in (a, b)$, since this would contradict Theorem 7.

If f has a derivative and is strictly increasing on (a, b), and if we have no other information concerning f, then all the same we have to conclude that $f'(x) \geqslant 0$ on (a, b), because a strictly increasing function can have a derivative equal to zero at individual points within (a, b). Such is, for instance, the function x^3 strictly increasing on $(-\infty, \infty)$ and having the derivative equal to zero for $x = 0$.

Remark 7. If a function is increasing at a point x_0, then it is not necessarily increasing in a certain neighbourhood of the point x_0.

This can be exemplified by the function

$$F(x) = \begin{cases} 0, & x = 0, \\ \frac{x}{2} - x^2 \sin\frac{1}{x}, & x \neq 0. \end{cases}$$

It is obvious that

$$F'(0) = \lim_{x \to 0} \frac{\frac{x}{2} - x^2 \sin\frac{1}{x}}{x} = \frac{1}{2}$$

and $F(x)$ is increasing at the point $x = 0$. But this function is not monotone, since the derivative $F'(x) = \frac{1}{2} - 2x \sin\frac{1}{x} + \cos\frac{1}{x}$ attains both positive and negative values in any small neighbourhood of zero (see Theorem 5). For $x_k = 1/k\pi$ ($k = 1, 2, \ldots$) it is equal to $3/2$ for even k's, and to $-1/2$ for odd k's.

Theorem 8. *If a function $f(x)$ is even (odd) and differentiable on a closed interval $[-a, a]$, then $f'(x)$ is an odd (even) function.*

Proof. Since $f(x) \equiv f(-x)$, $\forall x \in [-a, a]$, the derivatives of the left and right members also coincide: $f'(x) \equiv -f'(-x)$, i.e. $f'(x)$ is an odd function. (The same fact can be proved proceeding from the definition of the derivative.)

Sec. 4.13. Evaluation of Indeterminate Forms

The ratio $\frac{f(x)}{g(x)}$ will be said to represent the indeterminate form $\frac{0}{0}$ as $x \to a$, if $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$. To evaluate this indeterminate form means to find $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$, provided that it exists.

Theorem 1. *Let $f(x)$ and $g(x)$ be defined and differentiable in a neighbourhood of a point $x = a$ except, possibly, at the point a itself, $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$, $g(x)$ and $g'(x) \neq 0$ in this neighbourhood. Then if the limit $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ exists, the limit $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ also exists and is equal to the former*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \tag{1}$$

Proof. Let a be a finite number (for $a = \infty$ see Remark 3 below). We define the functions f and g at the point $x = a$, setting $f(a) = g(a) = 0$. Then these functions will be continuous at the point a. Consider the closed interval $[a, x]$, where $x > a$ or $x < a$ (see Remark 5 in the preced-

ing section). On $[a, x]$ the functions f and g are continuous, and on (a, x) they are differentiable, therefore, by Cauchy's theorem, there exists a point ξ such that

$$\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(\xi)}{g'(\xi)}\ (\xi\in(a,\ x))\ \text{or}\ \frac{f(x)}{g(x)}=\frac{f'(\xi)}{g'(\xi)}.$$

If $x\to a$, then also $\xi\to a$, therefore, by hypothesis, we have

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{\xi\to a}\frac{f'(\xi)}{g'(\xi)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \qquad (2)$$

provided that the limit in the right-hand side of the equality exists.

This proves our theorem.

Remark 1. If a right-hand limit in (1) does not exist, then a left-hand limit can exist.

Example. Since $\sin x\approx x$, we have

$$\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x}=\lim_{x\to 0}x\sin\frac{1}{x}=0.$$

But

$$\lim_{x\to 0}\frac{\left(x^2\sin\frac{1}{x}\right)'}{(\sin x)'}=\lim_{x\to 0}\frac{2x\sin\frac{1}{x}-\cos\frac{1}{x}}{\cos x}$$

is not existent.

Remark 2. If the expression $\frac{f'(x)}{g'(x)}$ represent the indeterminate form $\frac{0}{0}$ and the functions $f'(x)$, $g'(x)$ satisfy the condition of Theorem 1, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)}.$$

These equalities should be understood in the sense that if the third limit exists, then the first and the second limits also exist.

Theorem 2 $\left(\frac{\infty}{\infty}\right)$. *Let f and g be defined and differentiable in a neighbourhood of a point $x=a$,* $\lim\limits_{x\to a}f(x)=$

$= \lim_{x \to a} g(x) = \infty$, *$g(x)$ and $g'(x) \neq 0$ in this neighbourhood, then if*

$$\exists \lim_{x \to a} \frac{f'(x)}{g'(x)}, \text{ then } \exists \lim_{x \to a} \frac{f(x)}{g(x)}$$

and

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

We leave the proof of this theorem to the reader.

Remark 3. If $a = \infty$, then the substitution $x = 1/t$ reduces this case to $a = 0$:

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{t \to 0} \frac{f(1/t)}{g(1/t))} = \lim_{t \to 0} \frac{(f(1/t))'}{(g(1/t))'}$$
$$= \lim_{t \to 0} \frac{f'(1/t)(-1/t^2)}{g'(1/t)(-1/t^2)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}.$$

The rule stated in Theorems 1 and 2, according to which the computation of the limit of the ratio of functions can be reduced to the computation of the limit of the ratio of their derivatives is called *L'Hospital's rule* after G.F.A. L'Hospital* who stated this rule for some simpler cases; this rule was in fact known to Johann Bernoulli** before L'Hospital.

Example 1.

$$\lim_{x \to \infty} \frac{x^\alpha}{a^x} = 0, \ \forall \alpha > 0, \ a > 1.$$

Here we have an indeterminate form of the type $\frac{\infty}{\infty}$. Applying L'Hospital's rule k times ($k \geqslant \alpha$, for a natural α $k = \alpha$), we get

$$\lim_{x \to \infty} \frac{x^\alpha}{a^x} = \lim_{x \to \infty} \frac{cx^{\alpha - k}}{a^x (\ln a)^k} = 0.$$

---

* L'Hospital (also L'Hôpital, Lhospital), Guillaume François Antoine de (1661-1704). French mathematician.

** Bernoulli, Johann (or Jean or John) (1667-1748). Swiss mathematician.

12—01779

Example 2.

$$\lim_{x\to\infty} \frac{\ln x}{x^{\alpha}} = 0, \quad \forall \alpha > 0.$$

The functions x^{α} and $\ln x$ satisfy all the conditions of Theorem 2, therefore

$$\lim_{x\to\infty} \frac{\ln x}{x^{\alpha}} = \lim_{x\to\infty} \frac{1/x}{\alpha x^{\alpha-1}} = \lim_{x\to\infty} \frac{1}{\alpha x^{\alpha}} = 0.$$

Besides the two types of indeterminate forms treated above, we encounter the other types: $0\cdot\infty$, 0^0, ∞^0, $\infty-\infty$, 1^{∞}, whose evaluation is obvious. These are reduced to the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic transformations.

(a) The *indeterminate form* $0\cdot\infty$ ($f(x)\,g(x)$, $f(x)\to 0$, $g(x)\to\infty$ for $x\to a$). It is clear that

$$f(x)\,g(x) = \frac{f}{1/g}\left(\frac{0}{0}\right) \quad \text{or} \quad f\cdot g = \frac{g}{1/f}\left(\frac{\infty}{\infty}\right).$$

Example 3.

$$\lim_{x\to 0} x^{\alpha}\ln x = 0, \quad \forall\alpha > 0;$$

$$\lim_{x\to 0} x^{\alpha}\ln x = \lim_{x\to 0}\frac{\ln x}{x^{-\alpha}} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to 0}\frac{1/x}{-\alpha x^{-\alpha-1}}$$

$$= -\frac{1}{\alpha}\lim_{x\to 0} x^{\alpha} = 0.$$

(b) The *indeterminate forms* 1^{∞}, 0^0, ∞^0 for expressing f^g are reduced to the indeterminate form $0\cdot\infty$. According to the definition of this function $f^g = e^{g\ln f}$ $(f > 0)$.

If

$$\lim_{x\to a} g\ln f = k,$$

then

$$\lim_{x\to a} f^g = e^k.$$

(c) The *indeterminate form* $\infty - \infty$ ($f(x) - g(x)$, $f\to+\infty$, $g\to+\infty$ as $x\to a$). It is easy to see that

$$f - g = \frac{1}{\frac{1}{f}} - \frac{1}{\frac{1}{g}} = \frac{\frac{1}{g}-\frac{1}{f}}{\frac{1}{f}\cdot\frac{1}{g}}\left(\frac{0}{0}\right).$$

Sec. 4.14. Taylor's * Formula

Let us consider an arbitrary polynomial of degree n:

$$P_n(x) = b_0 + b_1 x + \ldots + b_n x^n = \sum_{k=0}^{n} b_k x^k,$$

where the constant numbers b_k are coefficients of the polynomial. Let x_0 be any fixed number. Setting $x = (x - x_0) + x_0$, we get

$$P_n(x) = \sum_{k=0}^{n} b_k [(x - x_0) + x_0]^k, \tag{1}$$

whence, on opening the square brackets and combining similar terms containing the same powers of $x - x_0$, we obtain the expression for $P_n(x)$ in the following form:

$$P_n(x) = a_0 + a_1(x - x_0) + \ldots + a_n(x - x_0)^n = \sum_{k=0}^{n} a_k (x - x_0)^k, \tag{2}$$

which is called the *expansion of the polynomial* $P_n(x)$ *in powers of* $(x - x_0)$. Here the numbers $a_0, a_1, \ldots, a_n$, dependent on b_i and x_0, are the coefficients of this expansion. For instance, $a_0 = b_0 + b_1 x_0 + \ldots + b_n x_0^n$. It is obvious from (1) that, in fact, $P_n(x)$ is independent of x_0.

Let us find in succession the derivatives of $P_n(x)$:

$$\left.\begin{aligned}
P_n'(x) &= a_1 + 2a_2(x - x_0) + \ldots + n a_n (x - x_0)^{n-1}, \\
P_n''(x) &= 1\cdot 2a_2 + 2\cdot 3a_3(x - x_0) \\
&\quad + \ldots + n(n-1) a_n (x - x_0)^{n-2}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
P_n^{(k)}(x) &= 1\cdot 2 \ldots k a_k + \ldots + n(n-1) \\
&\quad \ldots (n-k+1) a_n (x - x_0)^{n-k}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
P_n^{(n)}(x) &= 1\cdot 2 \ldots n a_n.
\end{aligned}\right\} \tag{3}$$

The derivatives of orders higher than n are equal to zero. Setting $x = x_0$ in formulas (2) and (3), we get

$$P_n(x_0) = a_0,\ P_n'(x_0) = a_1,\ P_n''(x_0) = 1\cdot 2a_2,\ \ldots,\ P_n^{(k)}(x_0) = k! a_k,\ \ldots,\ P_n^{(n)}(x_0) = n! a_n$$

* Taylor, Brook (1685-1731). English mathematician.

or

$$a_k = \frac{P_n^{(k)}(x_0)}{k!} \quad (k=0,\ 1,\ \ldots,\ n), \tag{4}$$

where we hold $0! = 1$, $P_n^{(0)}(x) = P_n(x)$.

Formulas (4) show that one and the same polynomial $P_n(x)$ of degree n can be expanded in powers of $x - x_0$ in a single way, i.e. if for all values of x

$$P_n(x) = \sum_{k=0}^{n} \beta_k (x-x_0)^k = \sum_{k=0}^{n} \beta'_k (x-x_0)^k,$$

where β_k and β'_k are constants, then $\beta_k = \beta'_k$ $(k = 0, 1, \ldots$ $\ldots, n)$. As a matter of fact, both numbers β_k and β'_k are computed using one and the same formula (4).

By virtue of (4), formula (2) can be rewritten as follows:

$$P_n(x) = P_n(x_0) + \frac{P'_n(x_0)}{1!}(x-x_0)$$

$$+ \ldots + \frac{P_n^{(n)}(x_0)}{n!}(x-x_0)^n = \sum_{k=0}^{n} \frac{P_n^{(k)}(x_0)}{k!}(x-x_0)^k. \tag{2'}$$

Formula (2′) is called *Taylor's formula, in powers of* $x - x_0$, *for the given polynomial* $P_n(x)$ *of the* nth *degree* or, simply, *Taylor's formula for* $P_n(x)$ *at the point* x_0. Note that the right-hand member of (2′) is, in fact, independent of x_0.

Example 1. Let $P_n(x) = (a + x)^n$ and $x_0 = 0$. Then, by (2′),

$$P_n(x) = \sum_{k=0}^{n} \frac{P_n^{(k)}(0)}{k!} x^k,$$

where in the given case

$$P_n^{(k)}(x) = n(n-1)\ldots(n-k+1)(a+x)^{n-k},$$
$$P_n^{(k)}(0) = n(n-1)\ldots(n-k+1)\,a^{n-k},$$

and we have obtained well-known *Newton's binomial formula*

$$(a+x)^n = \sum_{k=0}^{n} \frac{n(n-1)\ldots(n-k+1)}{k!} a^{n-k}x^k. \tag{5}$$

Let us now consider any function $f(x)$ which has continuous derivatives of all orders up to $(n+1)$th in some neighbourhood of the point x_0. Formally, we can write the polynomial

$$Q_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \tag{6}$$

which is called *Taylor's polynomial of degree n* or *n*th *Taylor's polynomial for the function f in powers of* $x-x_0$.

The polynomial $Q_n(x)$ coincides with the function $f(x)$ at the point x_0, but for all x's it is not equal to $f(x)$ (if $f(x)$ is not a polynomial of degree n). Besides,

$$Q_n'(x_0)=f'(x_0),\ \ldots,\ Q_n^{(n)}(x_0)=f^{(n)}(x_0). \tag{7}$$

We set

$$f(x)=Q_n(x)+r_n(x). \tag{8}$$

Formula (8) is called *Taylor's formula for the function* $f(x)$, and $r_n(x)$ is called the *remainder term of Taylor's formula or the* nth *remainder of Taylor's formula for the function f in powers of* $x-x_0$. The function $r_n(x)$ shows what error is introduced by replacing $f(x)$ by Taylor's polynomial (6).

Let us find the expression for $r_n(x)$ in terms of the derivative $f^{(n+1)}(x)$.

By (7) and (8), $r_n(x_0)=r_n'(x_0)=\ldots=r_n^{(n)}(x_0)=0$. Let us put $\varphi(x)=(x-x_0)^{n+1}$. It is clear that $\varphi(x_0)=\varphi'(x_0)=\ldots=\varphi^{(n)}(x_0)=0$. Applying Cauchy's theorem to the functions $r_n(x)$ and $\varphi(x)$, we shall have

$$\frac{r_n(x)}{\varphi(x)}=\frac{r_n(x)-r_n(x_0)}{\varphi(x)-\varphi(x_0)}=\frac{r_n'(x_1)}{\varphi'(x_1)}=\frac{r_n'(x_1)-r_n'(x_0)}{\varphi'(x_1)-\varphi'(x_0)}=\frac{r_n''(x_2)}{\varphi''(x_2)}$$

$$\ldots=\frac{r_n^{(n)}(x_n)}{\varphi^{(n)}(x_n)}=\frac{r_n^{(n)}(x_n)-r_n^{(n)}(x_0)}{\varphi^{(n)}(x_n)-\varphi^{(n)}(x_0)}=\frac{r_n^{(n+1)}(x_{n+1})}{\varphi^{(n+1)}(x_{n+1})}$$

($x_1\in(x_0,x)$ and $x_{k+1}\in(x_0,x_k)$, $k=1,2,\ldots n$).

But $\varphi^{(n+1)}(x)=(n+1)!$, $r_n^{(n+1)}(x)=f^{(n+1)}(x)-0=f^{(n+1)}(x)$.

Consequently,

$$r_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c), \tag{9}$$

where $c = x_{n+1}$ is a point lying between x_0 and x.

Thus, formula (8) can be written in the form

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}. \tag{8'}$$

Formula (8′) is called Taylor's formula with the remainder in *Lagrange's form*.

We have proved an important theorem.

Theorem 1. *If a function f has a continuous derivative $f^{n+1}(x)$ in a neighbourhood of a point x_0, then for any x from this neighbourhood there is a point $c \in (x_0, x)$ such that $f(x)$ can be written by formula* (8′).

Here c depends on x and n.

If the point $x_0 = 0$, then formula (8) is called *Maclaurin's formula*.

There are other forms of the remainder of Taylor's formula. Of great importance is *Cauchy's form*:

$$r_n(x) = \frac{(x-x_0)^{n+1}(1-\theta)^n}{n!} f^{(n+1)}(x_0 + \theta(x-x_0)), \tag{10}$$

where θ $(0 < \theta < 1)$ depends on n and x. This formula will be derived in Sec. 6.5.

Reducing the neighbourhood of the point x_0, we obtain that the derivative $f^{(n+1)}(x)$ is a continuous function of x on the closed interval $[x_0 - \delta, x_0 + \delta]$. But then it is bounded on this interval:

$$|f^{(n+1)}(x)| \leqslant M_n, \quad x_0 - \delta \leqslant x \leqslant x_0 + \delta \tag{11}$$

(see Sec. 3.5, Theorem 1). Here M_n is a positive number independent of the indicated x's, but, generally speaking, dependent on n. Then

$$|r_n(x)| \leqslant \frac{|f^{(n+1)}(c)|}{(n+1)!}|x-x_0|^{n+1} \leqslant \frac{M_n|x-x_0|^{n+1}}{(n+1)!}, \tag{12}$$
$$|x-x_0| < \delta.$$

Inequality (12) can be used for dual purpose: to investigate the behaviour of $r_n(x)$ with n fixed in the neigh-

bourhood of the point x_0 and to investigate the behaviour of $r_n(x)$ as $n \to \infty$.

From (12), for instance, it follows that for n fixed there holds the property

$$r_n(x) = o((x - x_0)^n), \quad x \to x_0, \tag{13}$$

showing that if $r_n(x)$ is divided by $(x - x_0)^n$, then the obtained quotient will keep tending to zero as $x \to x_0$.

By virtue of (13), it follows from (8'):

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \underset{x \to x_0}{o}((x - x_0)^n). \tag{14}$$

This formula is termed *Taylor's formula with remainder in Peano's* form*. It is adapted for studying the function f in the neighbourhood of the point x_0.

Theorem 2 (Uniqueness Theorem). *Let one and the same function f turn out, for different reasons, represented in a neighbourhood of a point x_0 in the form*

$$\left.\begin{aligned} f(x) &= a_0 + a_1(x - x_0) + \ldots + a_n(x - x_0)^n + \underset{x \to x_0}{o}((x - x_0)^n), \\ f(x) &= b_0 + b_1(x - x_0) + \ldots + b_n(x - x_0)^n + \underset{x \to x_0}{o}((x - x_0)^n). \end{aligned}\right\} \tag{15}$$

Then

$$a_k = b_k \quad (k = 0, 1, \ldots, n). \tag{16}$$

Proof. Equating the right-hand sides of (15) and passing to the limit as $x \to x_0$, we obtain the equality $a_0 = b_0$. Reducing now this equality by $(x - x_0)$ $(x \neq x_0)$ and passing to the limit once again as $x \to x_0$, we get $a_1 = b_1$. We proceed in this way till $a_n = b_n$ is obtained.

Example 2. We know that

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x} \quad (x \neq 1).$$

Therefore

$$\psi(x) = \frac{1}{1 - x} = \sum_{k=0}^{n} x^k + \frac{x^{n+1}}{1 - x} = \sum_{k=0}^{n} x^k + \underset{x \to 0}{o}(x^n). \tag{17}$$

---

* Peano, Giuseppe (1858-1932). Italian mathematician.

On the other hand, the function ψ has the derivatives of any order in a neighbourhood of the point $x = 0$, therefore it is representable by Taylor's formula with remainder in Peano's form:

$$\psi(x) = \sum_{k=0}^{n} \frac{\psi^{(k)}(0)}{k!} x^k + \underset{x \to 0}{o}(x^n). \tag{18}$$

Comparing formulas (17) and (18), by the uniqueness theorem, we get

$$1 = \frac{\psi^{(k)}(0)}{k!} \quad (k = 0, 1, \ldots, n). \tag{19}$$

The behaviour of the remainder of Taylor's formula as $n \to \infty$ is discussed in the next section.

Sec. 4.15. Taylor's Series

An expression

$$a_0 + a_1 + \ldots \tag{1}$$

or

$$\sum_{k=0}^{\infty} a_k, \tag{1'}$$

where a_k are numbers dependent on the index k, is called a (number) *series*. Finite sums

$$S_n = \sum_{k=0}^{n} a_k \quad (n = 0, 1, 2, \ldots)$$

are called *partial sums* of series (1) (or (1')). If there exists a finite limit

$$\lim_{n \to \infty} S_n = S, \tag{2}$$

then series (1) is said to be convergent to the number S, the latter being called the *sum* of a given series. We write

$$S = \sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \ldots$$

If the limit of partial sums S_n (as $n \to \infty$) of series (1) does not exist or is equal to ∞, then series (1) is said to be *divergent*.

Let now a function f have derivatives of any order in a neighbourhood of a point x_0. For such a function we can form a series of the following form:

$$f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots \tag{3}$$

or in a briefer notation

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k. \tag{3'}$$

Irrespective of whether this series is convergent or divergent, it is called *Taylor's series* for the function f in powers of $x - x_0$. If $x_0 = 0$, the corresponding series is sometimes called *Maclaurin's series.*

The case when Taylor's series of the function f in powers of $(x - x_0)$ is convergent in a neighbourhood of the point x_0 to that very function is of particular importance. In this case

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \quad x\in(x_0-\delta,\ x_0+\delta),$$

i.e. the function $f(x)$ is the sum of its Taylor's series in a certain neighbourhood of the point x_0. In this case the *function $f(x)$ is said to be expanded into Taylor's series in powers of $(x - x_0)$ convergent to $f(x)$.*

Theorem 1. *If a function f has derivatives of all orders on a closed interval $[x_0 - \delta, x_0 + \delta]$ and the remainder of its Taylor formula tends to zero as $n \to \infty$*

$$\lim_{n\to\infty} r_n(x) = 0, \quad x \in [x_0 - \delta,\ x_0 + \delta] \tag{4}$$

on this interval, then f is expanded into a Taylor's series convergent to f on this interval.

Proof. Let the function f have derivatives of all orders on a closed interval $[x_0 - \delta, x_0 + \delta]$. Then these derivatives are continuous on $[x_0 - \delta, x_0 + \delta]$, since if f has the derivative $f^{(k)}$ on $[x_0 - \delta, x_0 + \delta]$, then the derivative $f^{(k-1)}$ is continuous on this interval.

Therefore, Taylor's formula

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+r_n(x),$$

$$\forall n,\ x\in[x_0-\delta,\ x_0+\delta)],$$

makes sense for our function. Then, by virtue of (4),

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k=\lim_{n\to\infty}[f(x)-r_n(x)]$$

$$=f(x)-\lim_{n\to\infty}r_n(x)=f(x),$$

i.e. in this case Taylor's polynomial of the function $f(x)$ (in powers of $x-x_0$) tends to the function itself as $n\to\infty$:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k=f(x),\quad x\in[x_0-\delta,\ x_0+\delta]. \tag{5}$$

And this just means that the Taylor series of the function $f(x)$ converges on $[x_0-\delta,\ x_0+\delta]$ and its sum is equal to $f(x)$:

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k,\quad x\in[x_0-\delta,\ x_0+\delta].$$

The theorem has been proved.

The next theorem provides a simple sufficient test for the convergence of the remainder of Taylor's formula to zero.

Theorem 2. *If a function f has on a closed interval $[x_0-\delta,\ x_0+\delta]$ derivatives of all orders bounded by one and the same number ($|f^{(n)}(x)|\leqslant M$, $n=0, 1, 2, \ldots$ $\ldots$, $x_0-\delta\leqslant x\leqslant x_0+\delta$), then the remainder of its Taylor formula tends to zero on this interval as $n\to\infty$:*

$$\lim_{n\to\infty} r_n(x)=0. \tag{6}$$

Proof. Taking advantage of Lagrange's formula for the remainder, we get

$$|r_n(x)| = \frac{|x-x_0|^{n+1}}{(n+1)!} |f^{(n+1)}(c)| \leqslant \frac{M\cdot\delta^{n+1}}{(n+1)!}, \qquad (7)$$

($c \in (x_0, x)$, $M \geqslant |f^{(n+1)}(x)|$, $\forall n$ and $|x - x_0| < \delta$).

Since the right-hand member tends to zero as $n \to \infty$ (see Sec. 2.5, (5)), (6) holds true.

Sec. 4.16. Taylor's Formulas and Series for Most Important Elementary Functions

1. $f(x) = e^x$. This function is infinitely differentiable (has derivatives of any order) on $(-\infty, \infty)$. We have

$$f^{(k)}(x) = e^x, \quad f^{(k)}(0) = 1$$
$$(k = 0, 1, \ldots), \quad f^{(n+1)}(c) = e^c.$$

Therefore Taylor's formula for the function e^x with remainder in Lagrange's form is written as

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + r_n(x), \quad r_n(x) = \frac{e^c x^{n+1}}{(n+1)!}, \quad c \in (0, x), \qquad (1)$$

where x can be both positive and negative. On a closed interval $[-A, A]$, $A > 0$,

$$|r_n(x)| \leqslant \frac{e^A A^{n+1}}{(n+1)!} \to 0, \quad n \to \infty. \qquad (2)$$

This shows (see Theorem 1 in the preceding section) that the function e^x is expanded on $[-A, A]$ into Taylor's series in powers of x (i.e. into Maclaurin's series) convergent to e^x:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \qquad (3)$$

But since $A > 0$ is an arbitrary number, this equality holds throughout the entire real axis ($x \in (-\infty, \infty)$). In our case $|f^{(k)}(x)| = |e^x| \leqslant e^A$ ($k = 0, 1, 2, \ldots$) on the interval $[-A, A]$ and in order to obtain equality (3),

we could take advantage of Theorem 2 proved in the preceding section.

Let us compute the number e to within 0.001. We have (see (1))

$$e=\sum_{k=0}^{n}\frac{1}{k!}+r_n(1), \tag{4}$$

where

$$r_n(1)=\frac{e^c}{(n+1)!}, \quad 0<c<1; \tag{5}$$

n should be chosen so large that

$$r_n(1)=\frac{e^c}{(n+1)!}\leqslant 0.001 \quad (0<c<1).$$

Since $e^c<3$, it is sufficient to solve the inequality $3/(n+1)!\leqslant 0.001$. It is fulfilled for $n=6$. Consequently,

$$e\approx 2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{6!}=2.718$$

with an accuracy to 0.001.

Remark. Since $1<e^c<3$ for $0<c<1$, then for $n>2$ we have $e^c/(n+1)=\theta$, where $0<\theta<1$. Therefore, equality (4) can be written in the following form:

$$e=\sum_{k=0}^{n}\frac{1}{k!}+\frac{\theta}{n!}.$$

This formula was utilized (in Sec. 2.6, formula [3]) to prove the irrationality of the number e.

2. $y=\sin x$. This function has derivatives of all orders and

$$|(\sin x)^{(k)}|=\left|\sin\left(x+k\frac{\pi}{2}\right)\right|\leqslant 1, \ \forall k.$$

Therefore, by Theorem 2, the function $\sin x$ is expanded into Taylor's series in powers of x which is convergent to $\sin x$ on $(-\infty, \infty)$:

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

It should be taken into account that

$$(\sin x)^{(n)}\big|_{x=0}=\sin\frac{\pi n}{2}=\begin{cases}0 & \text{for } n=2k,\\ (-1)^k & \text{for } n=2k+1.\end{cases}$$

Taylor's formula for the function $\sin x$ in powers of x has the form

$$\sin x = x-\frac{x^3}{3!}+\ldots+(-1)^{\nu+1}\frac{x^{2\nu-1}}{(2\nu-1)!}+r_{2\nu}(x), \tag{6}$$

where

$$r_{2\nu}(x)=\frac{x^{2\nu+1}}{(2\nu+1)!}\sin\left(\theta x+(2\nu+1)\frac{\pi}{2}\right), \quad 0<\theta<1.$$

Hence it follows that

$$r_{2\nu}(x)=\underset{x\to 0}{o}(x^{2\nu})$$

and

$$\sin x = x-\frac{x^3}{3!}+\ldots+(-1)^{\nu+1}\frac{x^{2\nu-1}}{(2\nu-1)!}+\underset{x\to 0}{o}(x^{2\nu}).$$

3. $y = \cos x$. We can obtain in a similar way that

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}.$$

Example 1. Find $\lim\limits_{x\to 0}\frac{\sin x - x}{x^3}$.

We have

$$\sin x = x-\frac{x^3}{3!}+o(x^3), \tag{7}$$

therefore

$$\frac{\sin x - x}{x^3}=-\frac{1}{3!}+\frac{o(x^3)}{x^3}\underset{x\to 0}{=}-\frac{1}{3!}+\underset{x\to 0}{o}(1)\to-\frac{1}{3!},$$

i.e.

$$\lim_{x\to 0}\frac{\sin x - x}{x^3}=-\frac{1}{6}.$$

In fact, the remainder in (7) has the form $\underset{x\to 0}{o}(x^4)$, but for our purpose $\underset{x\to 0}{o}(x^3)$ is sufficient. It should be borne in mind

that if a certain function of x is $\underset{x\to 0}{o}(x^4)$, then it is also $\underset{x\to 0}{o}(x^3)$ (but, in general, not vice versa!).

4. The *function* $f(x) = \ln(1+x)$ is defined and infinitely differentiable for $x > -1$. Hence, Taylor's formula can be written for this function for any $n = 1, 2, \ldots$ and $x > -1$. Since

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}, \quad f^{(n)}(0) = (-1)^{n+1}(n-1)!,$$

Taylor's formula has the form

$$\ln(1+x) = x - \frac{x^2}{2} + \ldots + (-1)^{n+1}\frac{x^n}{n} + r_n(x).$$

Taking the remainder both in Lagrange's and in Cauchy's forms, we can show that

$$\lim_{n\to\infty} r_n(x) = 0 \quad \text{for} \quad -1 < x \leqslant 1.$$

Therefore the function $\ln(1+x)$ is expanded within the indicated interval into Taylor's series in powers of x:

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1}\frac{x^k}{k} \quad (-1 < x \leqslant 1).$$

5. The *function* $f(x) = (1+x)^m$. For this function

$$f^{(n)}(x) = m(m-1)\ldots(m-n+1)(1+x)^{m-n},$$
$$f^{(n)}(0) = m(m-1)\ldots(m-n+1).$$

Taylor's formula in powers of x has the form

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots + \frac{m(m-1)\ldots(m-n+1)}{n!}x^n + r_n(x).$$

We can prove that for any m

$$\lim_{n\to\infty} r_n(x) = 0 \quad (-1 < x < 1).$$

Therefore for any real m the function $(1+x)^m$ can be expanded into Taylor's series in powers of x

$$(1+x)^m=\sum_{k=0}^{\infty}\frac{m(m-1)\dots(m-k+1)}{k!}x^k \quad (-1<x<1). \tag{8}$$

If m is natural, then the function $(1+x)^m$ is a polynomial. In this case $r_n(x)\equiv 0$ for $n>m$ and the series on the right-hand side of (8) represents a finite sum, i.e. Taylor's polynomial (see Sec. 4.14).

Example 2. Evaluate the limit ($m\neq n$, $m\neq 0$, $n\neq 0$)

$$\lim_{x\to 0}\frac{\sqrt[m]{1+x}-\sqrt[n]{1+x}}{x}=\lim_{x\to 0}\frac{1+\frac{x}{m}+o(x)-\left(1+\frac{x}{n}+o(x)\right)}{x}$$

$$=\lim_{x\to 0}\frac{x\left(\frac{1}{m}-\frac{1}{n}\right)+o(x)}{x}=\lim_{x\to 0}\left[\left(\frac{1}{m}-\frac{1}{n}\right)+o(1)\right]$$

$$=\frac{1}{m}-\frac{1}{n}.$$

Example 3.

$$\lim_{x\to 0}\frac{\ln(1+x)-x(1+x)^{\alpha}}{x^2}=\lim_{x\to 0}\frac{x-\frac{x^2}{2}+o(x^2)-x(1+\alpha x+o(x))}{x^2}$$

$$=\lim_{x\to 0}\frac{-\left(\frac{1}{2}+\alpha\right)x^2+o(x^2)}{x^2}=\lim_{x\to 0}\left[-\frac{1}{2}-\alpha+o(1)\right]$$

$$=-\frac{1}{2}-\alpha.$$

Sec. 4.17. Local Extremum of a Function

The definition of local extremum was given at the beginning of Sec. 4.12. Obviously, it can also be formulated in the following way.

A function $y=f(x)$ attains its local maximum (minimum) at a point c if there is a positive number $\delta>0$ such that

its increment Δy *at the point* c *satisfies the inequality*

$$\Delta y = f(x) - f(c) \leqslant 0, \quad \forall x \in (c - \delta,\ c + \delta)$$

$(\Delta y = f(x) - f(c) \geqslant 0,\ \forall x \in (c - \delta,\ c + \delta))$.

According to Fermat's theorem (see Sec. 4.12), if a function f attains a local extremum at a point x_0 and there exists the derivative $f'(x_0)$ at this point, then this derivative is equal to zero:

$$f'(x_0) = 0.$$

By definition, a point x_0 is called a *stationary point of the function* f if the derivative of f exists at this point and is equal to zero ($f'(x_0) = 0$).

If a function f is defined on an open interval (a, b) and it is necessary to find all the points of its local extremum, then they, obviously, should be sought for among, firstly, its stationary points, that is among such points at which the derivative f' exists and equals zero and, secondly, among the points at which f has no derivative if such points actually exist. Stationary points are found from the equation

$$f'(x) = 0, \tag{1}$$

which should be solved. Of course, not any stationary point of the function f is a point of a local extremum of f.

Condition (1) is *necessary* for a differentiable function f to have a local extremum at point x, but not sufficient. For instance, $x = 0$ is a stationary point of the function x^3, but the function is increasing at this point.

It is also obvious that not every point at which f has no derivative is a point of a local extremum of f.

In any case, if it is already known that x_0 is a stationary point or a point at which the derivative of f does not exist, then we need some tests to identify whether this point is actually a point of a local extremum, and if it is, then whether it is a maximum or a minimum.

Given below are sufficient tests for a local extremum.

Theorem 1. *Let* x_0 *be a stationary point of a function* f *(i.e.* $f'(x_0) = 0$*) and* f *have a second continuous derivative in a neighbourhood of* x_0*. Then:*

if $f''(x_0) < 0$*, then* x_0 *is a point of a local maximum of* f*;*

and if $f''(x_0) > 0$, then x_0 is a point of a local minimum of f.

Proof. Let us expand f into Taylor's formula in powers of $(x - x_0)$ for $n = 1$. Since $f'(x_0) = 0$, Taylor's formula of the function f in a neighbourhood of the point x_0 has the form

$$f(x) = f(x_0) + \frac{(x-x_0)^2}{2} f''(c), \quad c \in (x_0, x), \tag{2}$$

where $x \gtrless x_0$ may occur.

Let $f''(x_0) < 0$. Since, by the hypothesis, f'' is continuous in a neighbourhood of x_0, there is $\delta > 0$ such that

$$f''(x) < 0, \quad \forall x \in (x_0 - \delta, x_0 + \delta).$$

But then the remainder term in formula (2)

$$\frac{(x-x_0)^2}{2} f''(c) \leqslant 0, \quad \forall x \in (x_0 - \delta, x_0 + \delta),$$

which shows that

$$\Delta y = f(x) - f(x_0) \leqslant 0, \quad \forall x \in (x_0 - \delta, x_0 + \delta),$$

i.e. at x_0 the function f has a local maximum.

Analogously, if $f''(x_0) > 0$, then $f''(x) > 0$ in some neighbourhood of x_0 and $f''(c) > 0$. Therefore the remainder term of formula (2) in a neighbourhood of x_0 is non-negative, and at the same time $\Delta y = f(x) - f(x_0) \geqslant 0$, i.e. f has a local minimum at x_0.

Example 1. $y = x^2 + 5$, $y' = 2x$, $x = 0$ is a stationary point; $y'' = 2 > 0$ for all values of x, hence, at the point $x = 0$ as well. Consequently, at the point $x = 0$ the given function has a local minimum.

Remark 1. If

$$f'(x_0) = 0 \quad \text{and} \quad f''(x_0) = 0, \tag{3}$$

then the function f may have and may not have an extremum at x_0. For instance, the functions x^3 and x^4 satisfy conditions (3) at the point $x_0 = 0$, but the first of them has no extremum at this point, while the second has an extremum, viz. a minimum.

Theorem 2. *Let $f'(x_0) = f''(x_0) = \dots = f^{(n)}(x_0) = 0$ and $f^{(n+1)}(x_0) \neq 0$ and be continuous in a neighbourhood of the point x_0, then:*

if $(n+1)$ is even and $f^{(n+1)}(x_0) < 0$, then f has a local maximum at x_0;

if $(n+1)$ is even and $f^{(n+1)}(x_0) > 0$, then f has a local minimum at x_0;

if $(n+1)$ is odd and $f^{(n+1)}(x_0) \neq 0$, then f, automatically, has no local extremum at x_0.

The proof of this theorem is based once again on application of Taylor's formula. We have

$$f(x) - f(x_0) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c), \quad c \in (x_0, x). \quad (4)$$

If $(n+1)$ is even, we reason just in the same way as in the case of formula (2). Let now $(n+1)$ be odd and let, as it was assumed, $f^{(n+1)}(x_0) \neq 0$. Owing to the continuity of $f^{(n+1)}$ in the neighbourhood of x_0, there exists an interval $(x_0 - \delta, x_0 + \delta)$, on which $f^{(n+1)}(x)$ retains the sign of $f^{(n+1)}(x_0)$. If x increases in the neighbourhood of x_0 from left to right, then, on passing through x_0, $(x - x_0)^{n+1}$ will change its sign, while the sign of $f^{(n+1)}(c)$ will remain unchanged. This shows that the right-hand member of (4) and, consequently, $\Delta y = f(x) - f(x_0)$ changes sign after x passes through x_0 and, therefore, an extremum at x_0 is impossible.

Theorem 3. *Let a function f be continuous on a closed interval $[x_0 - \delta, x_0 + \delta]$ and have the derivative $f'(x)$ separately on open intervals $(x_0 - \delta, x_0)$ and $(x_0, x_0 + \delta)$. Here*

$$f'(x) \geqslant 0 \quad (\leqslant 0) \quad on \quad (x_0 - \delta, x_0), \quad (5)$$

$$f'(x) \leqslant 0 \quad (\geqslant 0) \quad on \quad (x_0, x_0 + \delta). \quad (6)$$

Then x_0 is a point of local maximum (minimum) of the function f.

Here, it is not necessarily assumed that $f'(x_0)$ is existent.

Proof. From the continuity of f on $[x_0 - \delta, x_0]$ and property (5) it follows (see Theorem 5 proved in Sec. 4.12) that f is not decreasing (not increasing) on this interval,

and, consequently,

$$f(x_0) - f(x) \geqslant 0 \quad (\leqslant 0) \quad \text{for} \quad x \in [x_0 - \delta, x_0]. \tag{7}$$

And from the continuity of f on $[x_0, x_0 + \delta]$ and property (6) it follows (see the same theorem) that

$$f(x) - f(x_0) \leqslant 0 \quad (\geqslant 0) \quad \text{for} \quad x \in [x_0, x_0 + \delta]. \tag{8}$$

But then it follows from (7) and (8):

$$f(x) \leqslant f(x_0) \quad (f(x) \geqslant f(x_0)) \quad \text{for} \quad \forall x \in [x_0 - \delta, x_0 + \delta],$$

and we have proved Theorem 3.

Theorem 3 states that if the first derivative of the function f changes sign as x passes through the point x_0, then f has a minimum at the point x_0 (Fig. 52) if the sign changes (with an increase in x) from minus to plus, and a maximum (Fig. 53) if it changes from plus to minus. Here it is not essential whether $f'(x_0)$ exists or not, but it is necessary that f be continuous at the point x_0.

Example 2. The function $y = \frac{1}{1+x^2}$; $y' = \frac{-2x}{(1+x^2)^2}$.

We see that $y' > 0$ for $x < 0$, $y' < 0$ for $x > 0$ and, besides, y is continuous at the point $x = 0$, therefore, by Theorem 3, the function y has a local maximum at the point $x = 0$. The function has no other local extrema.

Example 3. The function $y = 2 - x^2\left(1 - \sin\frac{1}{x}\right)$ $(x \neq 0)$, $y(0) = 2$, is continuous at the point $x = 0$ and has a local maximum: $y(x) \leqslant 2 = y(0)$, $\forall x$. But we *fail to separate the neighbourhood of the point* $x = 0$ so that for $x < 0$ the function increases and for $x > 0$ decreases in it.

Indeed,

$$y' = -2x\left(1 - \sin\frac{1}{x}\right) - \cos\frac{1}{x} \quad (x \neq 0),$$

$$y'(0) = \lim_{x \to 0} \frac{2 - x^2\left(1 - \sin\frac{1}{x}\right) - 2}{x} = -\lim_{x \to 0} x\left(1 - \sin\frac{1}{x}\right) = 0.$$

For small x's the term $2x\left(1 - \sin\frac{1}{x}\right)$ is arbitrarily small, therefore the sign of the derivative y' depends on $\cos\frac{1}{x}$. As $x \to 0$,

$\cos \frac{1}{x}$ takes on the values ± 1 as many times as required. Hence, the function is oscillating in any neighbourhood of the point $x = 0$.

Theorem 4. *Let a function f satisfy the conditions $f'(x_0) = 0$ and $f''(x_0) > 0$ (<0). Then f has a local minimum (maximum) at the point x_0.*

Proof. Since

$$f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f'(x)}{x - x_0} > 0,$$

$\frac{f'(x)}{x - x_0} > 0$ in a sufficiently small neighbourhood, i.e. $f'(x) < 0$ for $x < x_0$ and $f'(x) > 0$ for $x > x_0$. By Theorem 3, we conclude that there is a local minimum at the point x_0. The case $f''(x_0) < 0$ is analysed in a similar way.

Remark 1. Theorem 4 contains Theorem 1 as a particular case, since it is not assumed in it that $f''(x)$ is continuous in a neighbourhood of the point x_0. It is only required the existence of $f''(x_0)$.

Sec. 4.18. Extreme Values of a Function on a Closed Interval

Let it be required to find the maximum (minimum) of a function f continuous on a closed interval $[a, b]$. The fact that f attains a maximum (minimum) on $[a, b]$ at a certain point $x_0 \in [a, b]$ was proved in Theorem 2 of Sec. 3.5.

Only three cases are possible: (1) $x_0 = a$, (2) $x_0 = b$, (3) $x_0 \in (a, b)$.

If $x_0 \in (a, b)$, then, according to what was said in the preceding section, x_0 will be a point of a local extremum and it should be sought for either among stationary points, or among points at which the derivative is not existent.

If the indicated points form a finite set $x_1, \ldots, x_m$, then

$$\max_{x \in [a, b]} f(x) = \max \{f(a), f(b), f(x_1), \ldots, f(x_m)\}$$

$$\left(\min_{x \in [a, b]} f(x) = \min \{f(a), f(b), f(x_1), \ldots, f(x_m)\}\right).$$

Note that there is no necessity to find out the character of stationary points if our task is only to find a maximum (minimum) of a function f on $[a, b]$.

Example 1. Find the greatest and the least values of the function

$$\psi(x) = \sin x + \cos x \quad \text{on} \quad [0, \pi].$$

We find the derivative: $\psi'(x) = \cos x - \sin x$ and equate it to zero:

$$\cos x - \sin x = 0.$$

This equation has a unique solution on $[0, \pi]$: $x = \pi/4$.

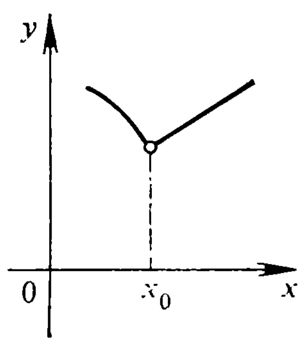

Fig. 52

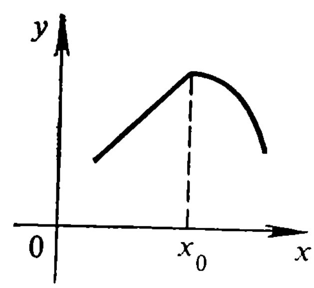

Fig. 53

Since $\psi(0) = 1$, $\psi(\pi/4) = \sqrt{2}$, $\psi(\pi) = -1$, we have

$$\max_{x \in [0, \pi]} \psi(x) = \sqrt{2}, \quad \min_{x \in [0, \pi]} \psi(x) = -1.$$

Example 2. Let an electric lamp be able to move along the vertical line OB (h-axis). On the plane perpendicular to OB let us take a point A (on the x-axis). Find the height at which the lamp should be suspended to ensure the best illumination at point A (Fig. 54).

Solution. Let us place the lamp at point B, and let $AB = r$, $OB = h$, $OA = a$, $\angle OAB = \varphi$. It is known that illumination I at point A is determined by the law $I = c\,\dfrac{\sin\varphi}{r^2}$, where c is a proportionality factor. Let

us take h for the argument of the function I. Since $r^2 = h^2 + a^2$, $\sin\varphi = \frac{h}{r}$, we have $I(h) = c\frac{h}{(h^2+a^2)^{3/2}}$.

According to the sense of our problem, $0 \leqslant h \leqslant \infty$. Let us find the greatest value of this function. $I(0) = I(\infty) = 0$*. Further

$$I'(h) = c\frac{a^2-2h^2}{(h^2+a^2)^{5/2}} = 0 \quad \text{for} \quad h = a/\sqrt{2}.$$

Since $I(a/\sqrt{2}) = 2c/3\sqrt{3}a^2 > 0$, the greatest value is attained by the function $I(h)$ at the point $h = a/\sqrt{2}$.

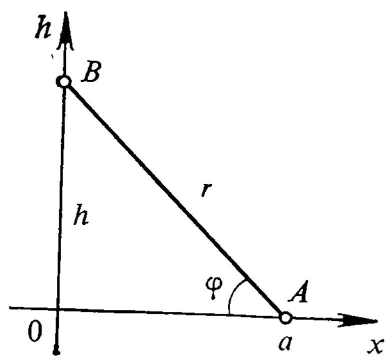

Fig. 54

Hence, the lamp should be suspended at a height $h = a/\sqrt{2}$.

Sec. 4.19. Convexity of a Curve. Points of Inflection

A *curve* $y = f(x)$ *is convex upward* (*downward*) *at a point* x_0** if there is a neighbourhood of x_0 such that the tangent line drawn to the curve at the point x_0 (that is, the tangent drawn through the point of the curve with abscissa x_0) lies above (below) the points of the curve for all abscissas x belonging to that neighbourhood (see Fig. 55; in the figure the curve is convex downward at the point x_1 and convex upward at the point x_2).

---

* In this case $I(\infty) = \lim\limits_{h\to+\infty} I(h)$.

** "Convex upward (downward)" is sometimes equivalently replaced by the expression "concave downward (upward)".

A point x_0 is called a *point of inflection of the curve* $y = f(x)$ if the moving point of the curve having abscissa x passes from one side of the tangent line (drawn through the point with abscissa x_0) to the other as x increases and passes through the value x_0 (in Fig. 55 the point x_3 is a point of inflection). In other words, there exists a sufficiently small $\delta > 0$ such that the curve lies on one side of the tangent line at the point x_0 for all $x \in (x_0 - \delta, x_0)$ and on the other side for all $x \in (x_0, x_0 + \delta)$.

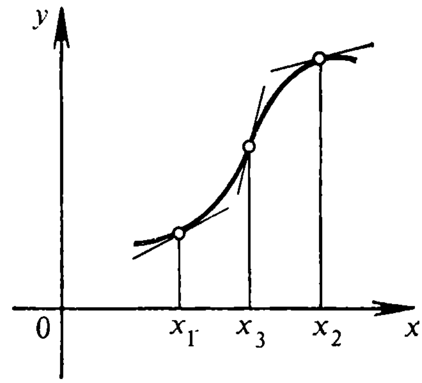

Fig. 55

These definitions describe some possible dispositions of a curve relative to its tangent line in a sufficiently small neighbourhood of the point of tangency. But it should be stressed that these definitions do not exhaust all possible configurations of a curve and its tangent line.

In the case of the function

$$f(x) = \begin{cases} 0, & x = 0, \\ x^2 \sin \frac{1}{x}, & x \neq 0, \end{cases}$$

the x-axis intersects its graph and touches the latter at the point $x = 0$, but $x = 0$ is not a point of inflection.

Theorem 1. *If a function f has the continuous second derivative f'' at a point x_0 and $f''(x_0) > 0$ (<0), then the curve $y = f(x)$ is convex downward (upward) at x_0.*

Proof. Let us represent f by Taylor's formula at the point $x = x_0$:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + r_1(x),$$

$$r_1(x) = \frac{(x-x_0)^2}{2} f''(x_0 + \theta(x - x_0)) \quad (0 < \theta < 1).$$

The equation of the tangent to our curve at the point having the abscissa x_0 will be written as

$$Y = f(x_0) + f'(x_0)(x - x_0).$$

Then the elevation of the curve f over its tangent drawn through the point x_0 is equal to

$$f(x) - Y = r_1(x).$$

Hence, the value of the remainder $r_1(x)$ is equal to the elevation of the curve f over its tangent line drawn through the point x_0. By the continuity of f'', we conclude that if $f''(x_0) > 0$, then $f''(x_0 + \theta(x - x_0)) > 0$ for all x's belonging to a sufficiently small neighbourhood of the point x_0 and therefore we, obviously, have $r_1(x) > 0$ for any value of x different from x_0 and belonging to this neighbourhood. Thus, the graph of the function lies above the tangent line and the curve is convex downward at x_0.

Analogously, if $f''(x_0) < 0$, then $r_1(x) < 0$ for any value of x different from x_0 and belonging to a certain neighbourhood of the point x_0, i.e. the graph of the function lies below the tangent line and the curve is convex upward at the point x_0.

Corollary. *If x_0 is a point of inflection of a curve $y = f(x)$ and at this point there exists the second derivative $f''(x_0)$, then the latter is necessarily equal to zero ($f''(x_0) = 0$).*

This is used in practice: when determining the points of inflection of a twice differentiable curve $y = f(x)$, they are sought for among the roots of the equation $f''(x) = 0$.

A sufficient condition for a point of inflection to exist is given by the following theorem.

Theorem 2. *If a function f is such that the derivative f''' is continuous at x_0 and $f''(x_0) = 0$ while $f'''(x_0) \neq 0$, then the curve $y = f(x)$ has a point of inflection for $x = x_0$.*

Proof. In this case we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + r_2(x),$$

$$r_2(x) = \frac{(x-x_0)^3}{3!} f'''(x + \theta(x - x_0)).$$

By the continuity of f''' at x_0 and by the fact that $f'''(x_0) \neq 0$, we conclude that $f'''(x_0 + \theta(x - x_0))$ retains sign in a neighbourhood of the point x_0 (i.e. the sign is the same to the right and to the left of the point x_0 within this neighbourhood). On the other hand, the factor

$(x - x_0)^3$ changes sign as x passes through x_0 and, together with it, the quantity $r_2(x)$ (equal to the elevation of the point of the curve over the tangent line drawn through the point x_0) also changes sign. This completes the proof of the theorem.

Now we state a more general theorem.

Theorem 3. *Let a function f possess the following properties:*

$$f''(x_0) = \ldots = f^{(n)}(x_0) = 0,$$

$f^{(n+1)}(x)$ is continuous at x_0 and $f^{(n+1)}(x_0) \neq 0$.

Then if n is an odd number, then the curve $y = f(x)$ is convex upward or downward at x_0 depending on whether $f^{(n+1)}(x_0) < 0$ or $f^{(n+1)}(x_0) > 0$, and if n is an even number, then x_0 is a point of inflection of the curve.

Proof. Here we use the expansion by Taylor's formula

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)} \times (x_0 + \theta(x - x_0)),$$

which is valid under the stated conditions.

In conclusion we note that when a function $y = f(x)$ has an infinite derivative f' equal to $+\infty$ or to $-\infty$ at a point x (see Figs. 40 and 41 in Sec. 4.2), we also say that the graph of the function has a point of inflection at x.

By definition, a curve $y = f(x)$ is called *convex upward (downward) on a closed interval* $[a, b]$ if any arc of this curve with end points at x_1, x_2 $(a \leqslant x_1 < x_2 \leqslant b)$ is situated not lower (not higher) than its chord (see Figs. 56 and 57).

Remark. If f is differentiable on $[a, b]$, then the above definition is equivalent to the following: a curve $y = f(x)$ is said to be *convex upward (downward) on a closed interval* $[a, b]$ if it is convex upward (downward) at every point x belonging to the open interval (a, b).

Theorem 4. *Let a function f be continuous on a closed interval $[a, b]$ and have a second derivative on the open interval (a, b).*

Then for the curve $y = f(x)$ to be convex upward (downward) on $[a, b]$, it is necessary and sufficient that the inequality $f''(x) \leqslant 0$ $(f''(x) \geqslant 0)$ be fulfilled for all $x \in (a, b)$.

We leave the proof of this theorem to the reader.

Example 1. The function $y = \sin x$ has a continuous first derivative and the second derivative $(\sin x)'' =$

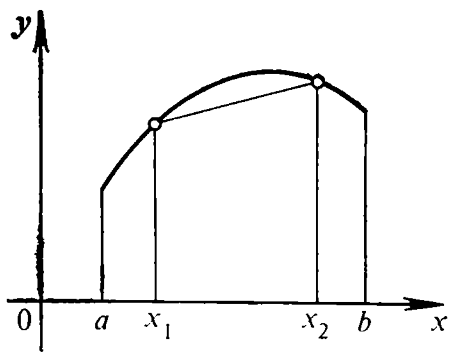

Fig. 56

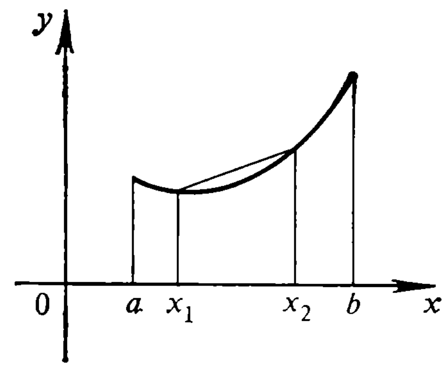

Fig. 57

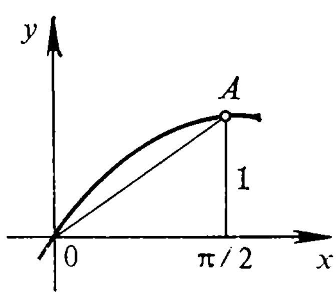

Fig. 58

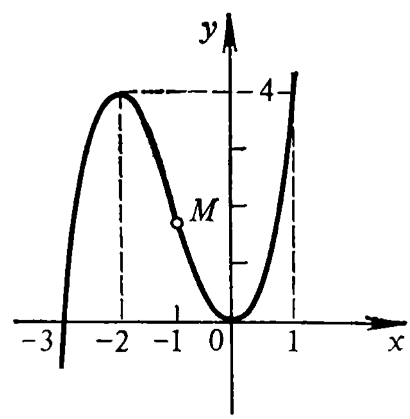

Fig. 59

$= -\sin x \leqslant 0$ on $[0, \pi/2]$. Therefore, the chord OA subtending the arc of the curve $y = \sin x$ on $[0, \pi/2]$ is below the sinusoid (Fig. 58). Since the equation of the chord is $y = (2/\pi)\, x$, we obtain the inequality

$$\frac{2}{\pi} x \leqslant \sin x, \quad 0 \leqslant x \leqslant \pi/2,$$

which is frequently used in mathematical analysis.

Example 2. $y = x^3 + 3x^2 = x^2(x + 3)$; $y' = 3x^2 + 6x$, $y' = 0$ for $x = 0$, $x = -2$; $y'' = 6x + 6$, $y''(0) =$

$= 6 > 0$, $y''(-2) = -6 < 0$, $y'' = 0$ for $x = -1$; $y''' = 6 \neq 0$. Since $y'''(x) = 6 \neq 0$, we have a point of inflection at $x = -1$. Further, $y''(x) > 0$ for $x > -1$, $y''(x) < 0$ for $x < -1$. Hence, the graph of the function (Fig. 59) is convex upward on $(-\infty, -1)$ and convex downward on $(-1, \infty)$; $x = 0$ is a point of minimum, $x = -2$ being a point of maximum.

Sec. 4.20. Asymptotes

The straight line $x = a$ is said to be a *vertical asymptote* of the graph of a continuous function $y = f(x)$ if at least one of the limits

$$\lim_{\substack{x \to a \\ x > a}} f(x) \quad \text{or} \quad \lim_{\substack{x \to a \\ x < a}} f(x)$$

is equal to ∞.

If a function $y = f(x)$ is defined for $x > M$ $(x < M)$, then the straight line $Y = kx + b$ is said to be an *inclined asymptote* of a continuous curve $y = f(x)$ for $x \to +\infty$ $(x \to -\infty)$ if $f(x) = kx + b + \alpha(x)$, where $\lim\limits_{\substack{x \to +\infty \\ (x \to -\infty)}} \alpha(x) = 0$ (i.e. $|f(x) - kx - b|$ is an infinitesimal function for $x \to +\infty$ $(x \to -\infty)$).

Example 1. $y = 1/x$ (Fig. 60); $x = 0$ is a vertical asymptote, since

$$\lim_{\substack{x \to 0 \\ x > 0}} \frac{1}{x} = +\infty, \quad \lim_{\substack{x \to 0 \\ x < 0}} \frac{1}{x} = -\infty.$$

Example 2. $y = x + \dfrac{\sin x}{x}$ $(x \neq 0)$.

Since $\lim\limits_{x \to \infty} \dfrac{\sin x}{x} = 0$, the straight line $Y = x$ (Fig. 61) is an inclined asymptote for $x \to +\infty$ (and for $x \to -\infty$).

Example 3. $y = \sqrt{x}$ $(x \geqslant 0)$. It is clear that $\sqrt{x} - kx - b$ does not tend to zero for any k and b as $x \to +\infty$, hence the function $y = \sqrt{x}$ has no inclined asymptotes.

Theorem. *In order for the graph of a function $y = f(x)$ to have an inclined asymptote for $x \to +\infty$ $(x \to -\infty)$,*

it is necessary and sufficient that there exist the finite limits

$$\lim_{x \to +\infty} \frac{f(x)}{x} = k, \qquad \lim_{x \to +\infty} [f(x) - kx] = b, \tag{1}$$

and then the straight line $Y = kx + b$ is an asymptote.

Proof. (1) Let the function $f(x)$ have an inclined asymp-

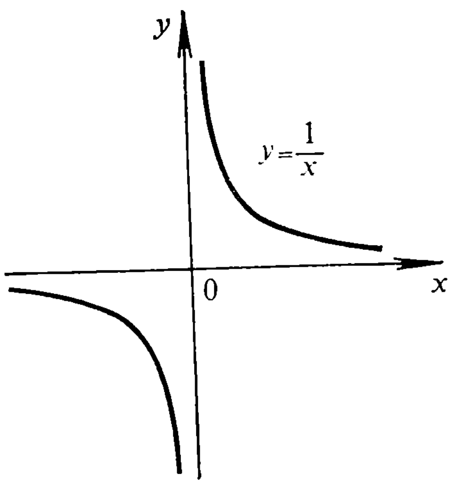

Fig. 60

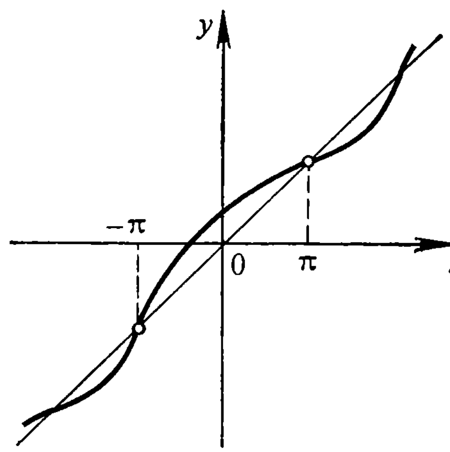

Fig. 61

tote $Y = kx + b$ for $x \to +\infty$. Then $f(x) = kx + b + \alpha(x)$, where $\alpha(x) \to 0$, $x \to +\infty$. Hence

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \left[k + \frac{b}{x} + \frac{\alpha(x)}{x}\right] = k,$$

$$\lim_{x \to +\infty} [f(x) - kx] = \lim_{x \to +\infty} [b + \alpha(x)] = b.$$

(2) Let the limits indicated in the theorem (as $x \to +\infty$) be existent. Then, from the second equality, by the definition of the limit, we have

$$f(x) - kx - b = \alpha(x), \text{ where } \alpha(x) \to 0 \text{ as } x \to +\infty,$$

i.e. $f(x) = kx + b + \alpha(x)$. Hence, the line $Y = kx + b$ is an inclined asymptote for $x \to +\infty$. For $x \to -\infty$ we reason in a similar way.

If $k = 0$, then we obtain a *horizontal asymptote*.

Remark. The existence of two *finite* limits (1) is essential, since for the function $y = \sqrt{x}$ $(x \geqslant 0)$,

$$\lim_{x\to+\infty} \frac{\sqrt{x}}{x} = 0 = k, \text{ but } \lim_{x\to+\infty} [\sqrt{x} - 0\cdot x] = \infty,$$

i.e. $b = \infty$, and this function has no asymptotes.

We can also give the following equivalent definition of an inclined asymptote.

If the distance $\rho(x)$ from the point $A(x, f(x))$ of a continuous curve $y = f(x)$ to a straight line $y = kx + b$ tends to zero as $x \to +\infty$ $(x \to -\infty)$, then the given straight line is called an inclined asymptote of this curve for $x \to +\infty$ $(x \to -\infty)$.

Indeed, it is known from analytical geometry that the distance from the point $(x, f(x))$ to the straight line $y = kx + b$ is expressed by the formula

$$\rho(x) = |f(x) - kx - b| / \sqrt{1+k^2},$$

whence from the fact that $|f(x) - kx - b| \to 0$ it follows that $\rho(x) \to 0$, and vice versa.

Example 4. Find out whether the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (|x| \geqslant a, \quad a \geqslant b > 0)$$

has asymptotes.

Solving the given equation with respect to y, we have

$$y = \pm \frac{b}{a} \sqrt{x^2 - a^2}.$$

Whence

$$\lim_{x\to+\infty} \frac{y}{x} = \pm \frac{b}{a} \lim_{x\to+\infty} \frac{\sqrt{x^2-a^2}}{x} = \pm \frac{b}{a}.$$

Further

$$\lim_{x\to+\infty} \left[y - \left(\pm \frac{b}{a} x\right)\right]$$

$$= \pm \frac{b}{a} \lim_{x\to+\infty} [\sqrt{x^2-a^2} - x] = \pm \frac{b}{a} \lim_{x\to+\infty} \frac{-a^2}{\sqrt{x^2-a^2}+x} = 0.$$

Thus, by the proved theorem, the straight lines

$$y = \pm \frac{b}{a} x$$

are asymptotes of our hyperbola, the plus sign referring to the upper half of the right-hand branch of the hyperbola, and the minus sign to its lower half.

By virtue of symmetry, it is clear that these straight lines are also asymptotes for $x \to -\infty$. In this case the

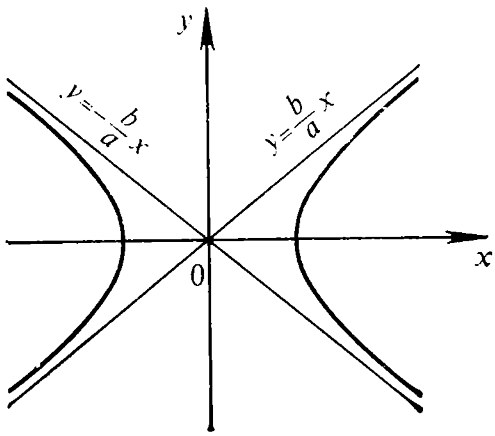

Fig. 62

plus sign corresponds to the portion of the hyperbola in question found in the third quadrant, while the minus sign to its portion located in the second quadrant (see Fig. 62).

Sec. 4.21. A Continuous and Smooth Curve

The equations

$$\left.\begin{aligned} x &= \varphi(t), \\ y &= \psi(t) \end{aligned}\right\} \quad (a < t < b), \tag{1}$$

where φ and ψ are continuous functions on (a, b), define a *continuous curve specified with the aid of the parameter* t, i.e. the locus of points $(\varphi(t), \psi(t))$, ordered by means of a parameter $t \in (a, b)$. With an increase in t, the point $(\varphi(t), \psi(t))$ moves in the plane. It may happen that to different t (t_1 and t_2) there corresponds one and the same point of the plane: $(\varphi(t_1), \psi(t_1)) = (\varphi(t_2), \psi(t_2))$.

Continuous curve (1) is called *smooth* if the functions $\varphi(t)$ and $\psi(t)$ have a continuous derivative on (a, b)

and the following inequality is fulfilled:

$$\varphi'(t)^2 + \psi'(t)^2 > 0, \quad \forall t \in (a, b). \tag{2}$$

Let us denote curve (1) by Γ and let $t_0 \in (a, b)$. By virtue of condition (2), one of the numbers $\varphi'(t_0)$, $\psi'(t_0)$ is different from zero. For the sake of definiteness, let $\varphi'(t_0) \neq 0$. But then, by the continuity of $\varphi'(t)$, there exists an open interval $(t_0 - \delta, t_0 + \delta)$ on which $\varphi'(t)$ retains the sign of $\varphi'(t_0)$. Consequently, $\varphi(t)$ is strictly monotone on $[t_0 - \delta, t_0 + \delta]$ and, besides, as we know, is continuously differentiable. In such case the function $x = \varphi(t)$ has an inverse

$$t = \varphi^{-1}(x) = g(x), \quad x \in (c, d), \tag{3}$$

which is strictly monotone and continuously differentiable on some interval (c, d), viz. in a neighbourhood of the point $x_0 = \varphi(t_0)$.

Substituting the expression for t into the second equation of (1), we obtain that the piece γ of our curve Γ, corresponding to the interval $(t_0 - \delta, t_0 + \delta)$, is described by a continuously differentiable function (see Sec. 4.4, Theorem 1)

$$y = F(x) = \psi[\varphi^{-1}(x)], \quad x \in (c, d), \tag{4}$$

and therefore at any point of γ there exists a tangent line not parallel to the y-axis. Obviously, the points of γ are projected on the x-axis one-to-one.

If now $\psi'(t_0) \neq 0$, then, reasoning in a similar way, we obtain that the piece γ_1 of the curve Γ corresponding to a sufficiently small interval $(t_0 - \delta, t_0 + \delta)$ is described by a continuously differentiable function

$$x = \Phi(y) = \varphi[\psi^{-1}(y)], \quad y \in (c_1, d_1). \tag{5}$$

Hence it follows that in this case there also exists a tangent line at any point of γ_1, but now it is not parallel to the x-axis.

Thus, at any point of a smooth curve Γ there exists a tangent line which can be parallel to one of the coordinate axes.

Example. The equations

$$\left.\begin{aligned} x &= a\cos t, \\ y &= b\sin t \end{aligned}\right\} \quad (-\infty < t < \infty)$$

represent parametrically an ellipse with the semiaxes a and b (Fig. 63).

This is a smooth curve, since the functions $x = a\cos t$

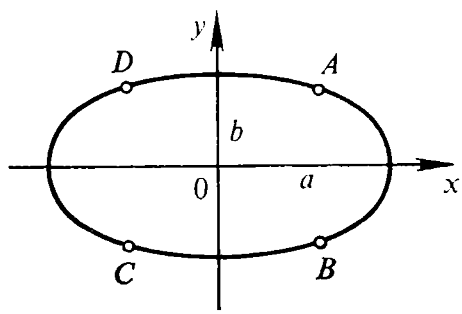

Fig. 63

and $y = b\sin t$ have continuous derivatives not equal to zero simultaneously

$$(x'(t))^2 + (y'(t))^2 = (a\sin t)^2 + (b\cos t)^2$$
$$\geqslant b^2(\sin^2 t + \cos^2 t) = b^2 > 0 \quad (0 < b \leqslant a).$$

The points A, B, C, D (see Fig. 63) divide the ellipse into four smooth pieces each of which is projected one-to-one either on the x-axis or on the y-axis.

Sec. 4.22. General Scheme for Constructing the Graph of a Function

If it is required to conceive a general idea of the graph of a function $y = f(x)$, then the following directions can be helpful.

1. Find the domain Ω of values of x, where the function f is defined.

2. Find the points $x_1, x_2, \ldots$, at which $f'(x) = 0$ or the derivative is not existent, in particular, equal to ∞. Compute the values of f at these points: $f(x_1), f(x_2), \ldots$, provided that they exist, and determine whether they are points of maximum or minimum. If f is not defined

at some of the points x_k, then it is of importance to know the limits $f(x_k - 0)$, $f(x_k + 0)$, and also to determine the limits

$$f(-\infty) = \lim_{x \to -\infty} f(x), \quad f(+\infty) = \lim_{x \to +\infty} f(x),$$

if they have sense.

3. The domain Ω is separated by the points x_k into intervals (a, b) on each of which $f'(x) \neq 0$. Among them there may be infinite intervals (of the form $(-\infty, c)$ or (d, ∞)).

We shall hold that the derivative $f'(x)$ is continuous on each of such intervals (a, b). Then $f'(x)$ retains sign on (a, b). It is important to find out this sign to know whether the function f increases or decreases on (a, b).

4. Mark on each interval (a, b) the points

$$x_{k,1}, x_{k,2}, \ldots \quad (k = 0, 1, 2, \ldots),$$

where $f''(x) = 0$, and determine the corresponding values of the function

$$f(x_{k,1}), f(x_{k,2}), \ldots.$$

These points may turn out to be the points of inflection of the curve $y = f(x)$. These points, in their turn, divide (a, b) into subintervals on which the second derivative (if it exists) retains sign.

The investigation of the sign of $f''(x)$ makes it possible to find out the direction of the convexity of the curve (upward or downward).

5. Whenever it is possible, solve the equation $f(x) = 0$ and determine the intervals on which f retains sign ($f(x) > 0$ or $f(x) < 0$).

6. Find out whether the curve has asymptotes or, in other words, find the limits

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = k, \quad \lim_{x \to \pm\infty} [f(x) - kx] = b,$$

provided that they exist.

14—01779

The results of the above investigations are then tabulated as it is done below:

x	$(-\infty, 0)$	0	$\left(0, \frac{1}{2}\right)$	$\frac{1}{2}$	$\left(\frac{1}{2}, 1\right)$	1	$(1, \infty)$
y'	>0	0	<0	$-\frac{3}{2}$	<0	0	>0
y	increases, no asymptotes	0	decreases	$-\frac{1}{2}$	decreases	-1	increases, asymptote $y=x-3$
y''	<0	<0	<0	0	>0	>0	>0
y	convex upward	max	convex upward	inflection	convex downward	min	convex downward

Constructed from this table, the graph of the function $y = f(x)$ has the form shown in Fig. 64.

Of course, this graph gives us exact values of f only at three points ($x = 0, 1/2, 1$), all the remaining values of f being represented approximately (that is, estimated by eye), but still it conveys the general idea about the behaviour of the function under consideration. Suppose we want to tabulate it in more detail, say, to compute it on a certain interval for the values of x spaced at 0.001. In this event we would have to take advantage of some computing device (calculator, computer, and the like). But, nevertheless, prior to utilizing computing devices it is advisable to draft a schematic graph (as is shown in Fig. 64) so as to grasp the general character of the curve.

Example. Construct the curve represented parametrically:

$$\left.\begin{array}{l} x = te^{t}, \\ y = te^{-t} \end{array}\right\} \quad (-\infty < t < \infty). \tag{1}$$

Solution. Let us first construct the graph of the function $x = te^t$, defined throughout the entire axis. It is un-

bounded, continuous, and differentiable on $(-\infty, \infty)$; $x > 0$ for $t > 0$; $x < 0$ for $t < 0$; $x = 0$ for $t = 0$. Then, $x' = (1 + t)\, e^t$. The equation $x'(t) = 0$ has the unique root $t = -1$. In this case, obviously, $x' > 0$ for $t > -1$; $x' < 0$ for $t < -1$. Hence, the function $x(t)$ increases

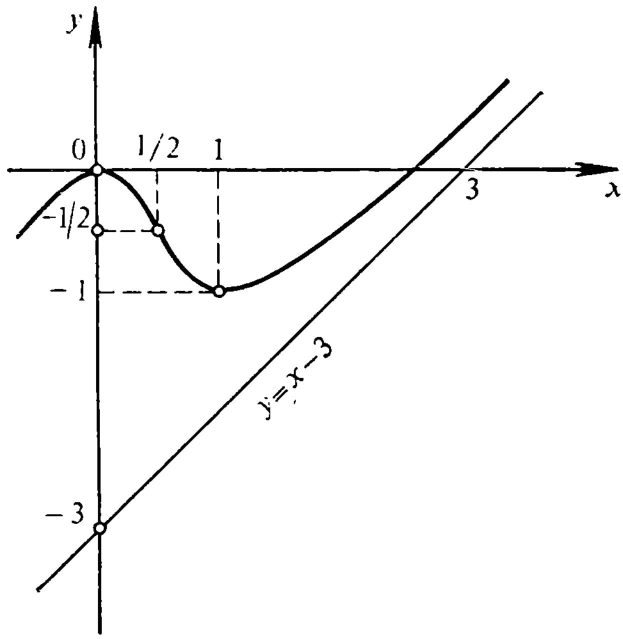

Fig. 64

for $t > -1$ and decreases for $t < -1$. At the point $t = -1$ the function $x(t)$ has a local minimum $x(-1) = -e^{-1}$. In fact, this is, obviously, a minimum on $(-\infty, \infty)$.

Let us now investigate the given function for convexity: $x'' = (2 + t)\, e^t$; $x'' > 0$ for $t > -2$; $x'' < 0$ for $t < -2$; $x''(-2) = 0$. Hence, we conclude that the graph is convex upward on $(-\infty, -2)$ and downward on $(-2, \infty)$, $t = -2$ being a point of inflection.

Further,

$$\lim_{t\to-\infty} \frac{te^t}{t} = 0, \quad \lim_{t\to-\infty} [te^t - 0] = 0,$$

i.e. $x = 0$ is a horizontal asymptote.

All this determines the graph of the function $x = te^t$ whose shape is shown in Fig. 65. The domain of values of the function $X = [-e^{-1}, \infty)$.

Now we are going to construct the graph of the function $y = te^{-t}$ (Fig. 66) which is done in a similar way. The domain of values of this function $Y = (-\infty, e^{-1})$. On $(-\infty, 1)$ the function $y = te^{-t}$ is strictly increasing

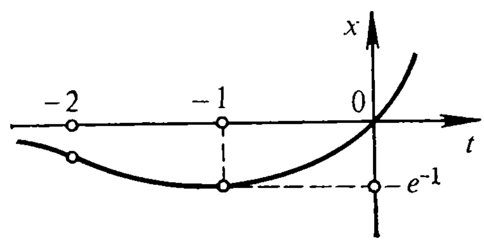

Fig. 65

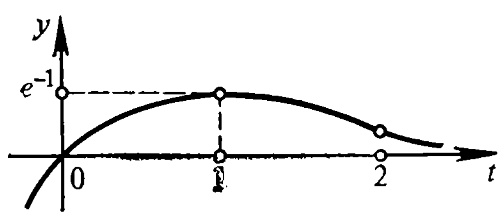

Fig. 66

from $-\infty$ to $y = e^{-1}$, at the point $t = 1$ it attains both a local maximum and maximum on $(-\infty, \infty)$. On the interval $(1, \infty)$ it strictly decreases to zero for $t \to +\infty$ and thus has an asymptote $y = 0$ for $t \to +\infty$ We must also mention the point $t = 2$ as a point of inflection of the curve, the latter being convex upward on $(-\infty, 2)$ and downward on $(2, \infty)$.

Now we pass to a more difficult problem, that is, to drawing a schematic graph of curve (1). Let us denote this curve by Γ. The functions determining Γ are infinitely continuously differentiable. We shall make use of the fact that these functions are twice continuously differentiable. Note that Γ is a smooth curve because the derivatives (with respect to t) of the functions $x = \varphi(t) = te^t$ and $y = \psi(t) = te^{-t}$ are not equal both to zero.

Denoting by Γ_1 and Γ_2 the branches of Γ on which $x'_t < 0$ and $x'_t > 0$, respectively, we obtain the following (see Figs. 65 and 66):

Γ_1 corresponds to the variation of $t \in (-\infty, -1)$,

Γ_2 corresponds to the variation of $t \in (-1, \infty)$.

On Γ_1 the function $x = \varphi(t)$ strictly decreases from $\varphi(-\infty) = 0$ to $\varphi(-1) = -e^{-1}$ and it can be converted, while the function $y = \psi(t)$ strictly increases from $\psi(-\infty) = -\infty$ to $\psi(-1) = -e$. Hence it follows that the branch Γ_1 is described by the explicit function

$$y = \psi[\varphi^{-1}(x)], \quad x \in (-e^{-1}, 0).$$

It is represented in Fig. 67 below the point A. When t increases from $-\infty$ to -1, the abscissa x of a point on Γ_1 decreases from 0 to $-e^{-1}$, while the ordinate y increases from $-\infty$ to $-e$. Since $x'(-1) = 0$ and $y'(-1) \neq 0$, the tangent line at the point A is parallel to the y-axis. Moreover, Γ is situated on the right of the tangent line; in fact, it is seen from Fig. 65 that all the points belonging to Γ have the abscissa $x \geqslant -e^{-1}$.

At any point t of the curve Γ, other than A, i.e. for $t \neq -1$ the derivative $x'(t) \neq 0$ and

$$y'_x = \frac{y'_t}{x'_t} = \frac{1-t}{1+t} e^{-2t}, \tag{2}$$

$$y''_x = \frac{dy'_x}{dx} = \frac{\left(\frac{1-t}{1+t} e^{-2t}\right)'}{(1+t)e^t} = 2\frac{t^2-2}{(1+t)^3} e^{-3t}. \tag{3}$$

Hence

$$y''_x \big|_{t=\pm\sqrt{2}} = 0. \tag{4}$$

We are now interested in the value $t = -\sqrt{2}$, to which there corresponds the point $B = \left(-\sqrt{2}e^{-\sqrt{2}}, -\sqrt{2}e^{\sqrt{2}}\right) \in \Gamma_1$.

It is seen from (3) that if $t < -\sqrt{2}$ (i.e. lies on the branch Γ_1 below the point B), then $y''_x < 0$ and Γ_1 is convex upward. And if $-\sqrt{2} < t \leqslant -1$ (i.e. on the arc $\overset{\frown}{AB}$), then $y''_x > 0$ and Γ_1 is convex downward. Hence, B is a point of inflection of Γ_1.

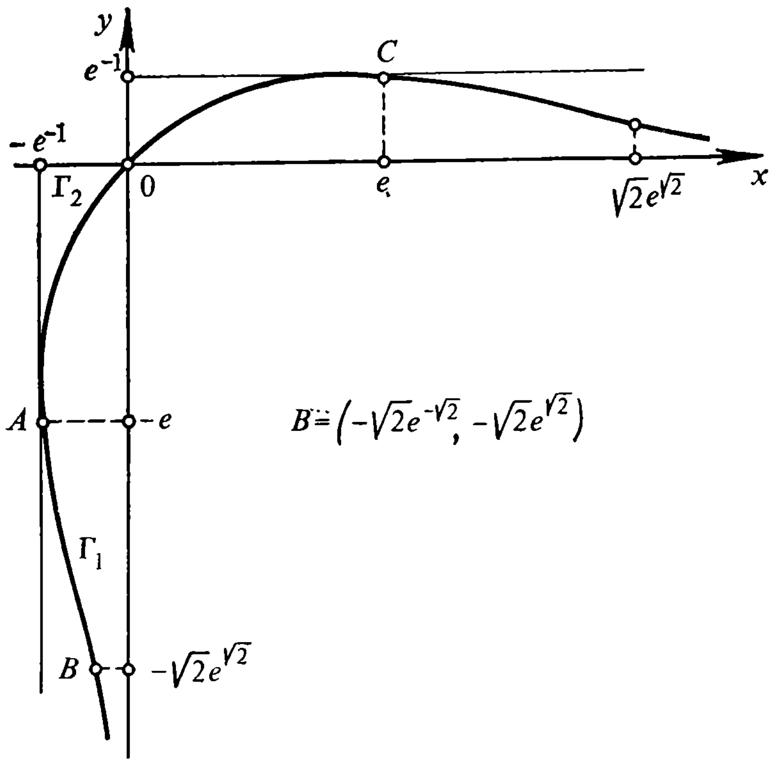

Fig. 67

Now we pass to Γ_2 $(-1 < t < \infty)$. As is seen from Figs. 65 and 66, on the interval $-1 < t < 1$ the functions $x = \varphi(t)$ and $y = \psi(t)$ strictly increase, but then the function of x

$$y = \psi[\varphi^{-1}(x)] \quad (-e^{-1} < x < e)$$

also strictly increases, and its graph on this interval is convex upward (see (3)). This is represented by the arc $\breve{AC} \subset \Gamma_2$. As to the point C, we have $y'_x = 0$ $\left(y'_x(e) = \frac{y'(1)}{x'(1)} = \frac{0}{x'(1)} = 0\right)$, and, since the graph is convex upward at this point, we conclude that C is a point of a local maximum of the function $y(x)$. For $x > e$ (i.e. for $t > 1$) $x(t)$ increases, while $y(t)$ decreases tending to zero. This shows that, decreasing, $y(x) \underset{x \to +\infty}{\longrightarrow} 0$. In this case $x(\sqrt{2}) = \sqrt{2}e^{\sqrt{2}}$ is a point of inflection of the graph of $y(x)$. To the left of this point the graph is convex upward, and to the right of it, downward (see (3)).

Sec. 4.23. Vector Function. The Vectors of the Tangent and Normal

Let us be given a rectangular coordinate system in the plane (x, y). The equations

$$\left.\begin{aligned} x &= x(t), \\ y &= y(t) \end{aligned}\right\} \quad (a < t < b), \tag{1}$$

where $x(t)$ and $y(t)$ are continuous functions on an open interval (a, b) which determine a *continuous curve* Γ, i.e. a geometrical set of points $(x(t), y(t))$ of the plane, where $t \in (a, b)$. The curve Γ is also said to be specified with the aid of the parameter t. Its equation can be given in the vector form

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \quad (a < t < b), \tag{1'}$$

where $\mathbf{i}$, $\mathbf{j}$ are unit vectors of the x- and y-axes, respectively, and $\mathbf{r} = \mathbf{r}(t)$ is the radius vector of the point on Γ

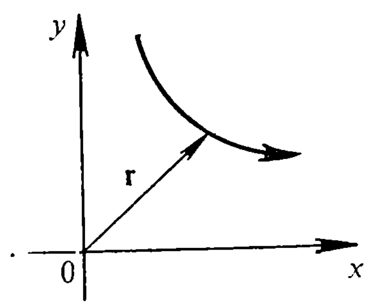

Fig. 68

corresponding to the value t of the parameter (Fig. 68).

The vector $\mathbf{r}(t)$ is called a *vector function* (defined for $t \in (a, b)$).

In this connection the curve Γ is said to be the *hodograph* of the vector function $\mathbf{r}(t)$ which is defined as the locus of the termini of the vectors $\mathbf{r}(t)$ emanating from the origin O.

A curve Γ is said to be *smooth* on (a, b) if the functions $x(t)$ and $y(t)$ have continuous derivatives on (a, b) not equal both to zero.

If t is assigned an increment Δt, then the vector $\mathbf{r}$ will receive the increment (see Fig. 69)

$$\begin{aligned}\Delta\mathbf{r} &= \mathbf{r}(t+\Delta t)-\mathbf{r}(t)\\ &= [x(t+\Delta t)-x(t)]\,\mathbf{i}+[y(t+\Delta t)-y(t)]\,\mathbf{j}\\ &= \Delta x\,\mathbf{i}+\Delta y\,\mathbf{j},\end{aligned}$$

whence, dividing by the scalar Δt, we get

$$\frac{\Delta\mathbf{r}}{\Delta t}=\frac{\Delta x}{\Delta t}\mathbf{i}+\frac{\Delta y}{\Delta t}\mathbf{j}.$$

For a smooth curve

$$\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}=x', \quad \lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}=y'.$$

The vector $x'\mathbf{i}+y'\mathbf{j}$ is called the *derivative of* $\mathbf{r}$ (at the point t); the latter is denoted as

$$\dot{\mathbf{r}}=x'\mathbf{i}+y'\mathbf{j}.$$

The derivative $\dot{\mathbf{r}}$ can also be defined as such a vector for which

$$\left|\frac{\Delta\mathbf{r}}{\Delta t}-\dot{\mathbf{r}}\right|\to 0, \quad \Delta t\to 0.$$

Indeed,

$$\left|\frac{\Delta\mathbf{r}}{\Delta t}-\dot{\mathbf{r}}\right|^2=\left(\frac{\Delta x}{\Delta t}-x'\right)^2+\left(\frac{\Delta y}{\Delta t}-y'\right)^2\to 0, \quad \Delta t\to 0.$$

We write

$$\dot{\mathbf{r}}=\lim_{\Delta t\to 0}\frac{\Delta\mathbf{r}}{\Delta t}$$

and say that the vector $\dot{\mathbf{r}}$ is the *limit of the vector* $\Delta\mathbf{r}/\Delta t$ *as* $\Delta t\to 0$. As is seen from Fig. 69, the vector $\dot{\mathbf{r}}$ is directed along the tangent to Γ at the point t towards increasing values of the parameter t.

The vector $\dot{\mathbf{r}}$ is called a *tangent vector* to Γ. Its length is equal to

$$|\dot{\mathbf{r}}|=\sqrt{x'^2+y'^2}.$$

The *unit vector of the tangent* is defined as

$$\boldsymbol{\tau} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j} \quad (|\dot{\mathbf{r}}| > 0),$$

$$\cos\alpha = \frac{x'}{\sqrt{x'^2+y'^2}}, \quad \sin\alpha = \frac{y'}{\sqrt{x'^2+y'^2}}, \tag{2}$$

where α is the angle between $\boldsymbol{\tau}$ and the positive direction of the x-axis.

The *unit normal vector* to Γ, i.e. the unit vector per-

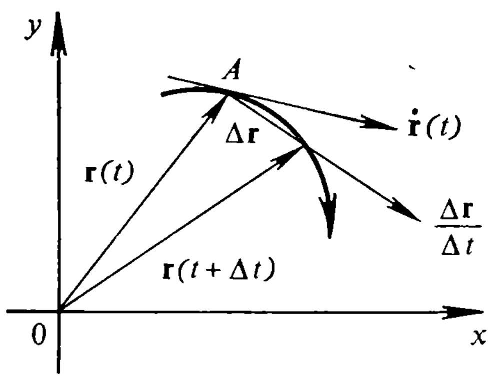

Fig. 69

pendicular to $\boldsymbol{\tau}$ is defined by the equality

$$\boldsymbol{\nu} = (\nu_1, \nu_2),$$

$$\nu_1 = \mp\sin\alpha, \quad \nu_2 = \pm\cos\alpha \tag{3}$$

or

$$\nu_1 = \mp\frac{y'}{\sqrt{x'^2+y'^2}}, \quad \nu_2 = \pm\frac{x'}{\sqrt{x'^2+y'^2}}. \tag{3'}$$

The determinant

$$\begin{vmatrix} \tau_1 & \tau_2 \\ \nu_1 & \nu_2 \end{vmatrix} = \begin{vmatrix} \cos\alpha & \sin\alpha \\ \mp\sin\alpha & \pm\cos\alpha \end{vmatrix} = \pm 1.$$

The upper signs correspond to the case when a pair of vectors $(\boldsymbol{\tau}, \boldsymbol{\nu})$ is oriented in the same way as the axes $(\mathbf{i}, \mathbf{j})$ (Fig. 70), while the lower ones when the pair $(\boldsymbol{\tau}, \boldsymbol{\nu})$ is oriented in an opposite way (Fig. 71).

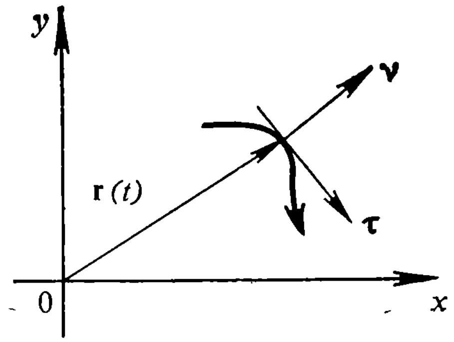

Fig. 70

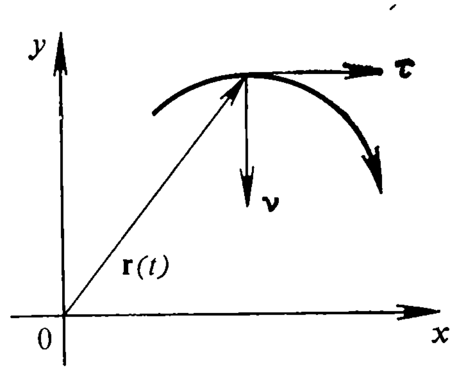

Fig. 71

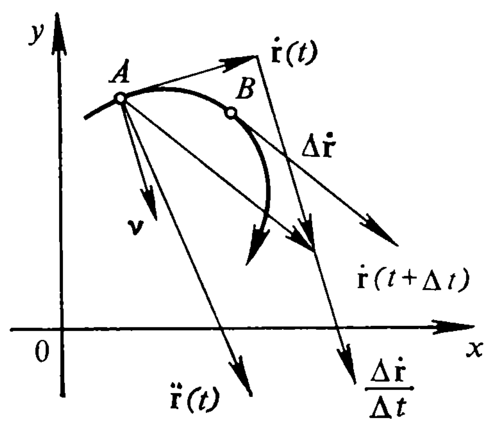

Fig. 72

The second derivative of the vector function $\mathbf{r}(t)$ (see (1′)) is defined as the limit

$$\ddot{\mathbf{r}}(t)=\lim_{\Delta t\to 0}\frac{\dot{\mathbf{r}}(t+\Delta t)-\dot{\mathbf{r}}(t)}{\Delta t}=x''(t)\,\mathbf{i}+y''(t)\,\mathbf{j}.$$

Figure 72 represents a curve Γ whose point A corresponds to the value t and B to the value $t+\Delta t$. Applied to these points are tangent vectors $\dot{\mathbf{r}}(t)$ and $\dot{\mathbf{r}}(t+\Delta t)$. The second vector is translated to the point A. Shown in the figure are: the difference $\Delta\dot{\mathbf{r}}=\dot{\mathbf{r}}(t+\Delta t)-\dot{\mathbf{r}}(t)$, the vector $\Delta\dot{\mathbf{r}}/\Delta t$ having the same direction as $\Delta\dot{\mathbf{r}}$, and finally, the limit vector $\ddot{\mathbf{r}}=\ddot{\mathbf{r}}(t)$. *The vector* $\ddot{\mathbf{r}}$ *is in the direction of the concavity of* Γ. The exact meaning of these

words is as follows: *the vector* $\ddot{\mathbf{r}}$ *forms an acute angle with the vector* $\boldsymbol{\nu}$ *of the normal to* Γ *drawn in the direction of the concavity of* Γ.

Example. The following is the vector-form equation of the ellipse (see Sec. 4.21)

$$\mathbf{r} = \mathbf{i}a \cos t + \mathbf{j}b \sin t \quad (-\infty < t < \infty).$$

Then, the tangent vector will be

$$\dot{\mathbf{r}} = -\mathbf{i}a \sin t + \mathbf{j}b \cos t,$$

and the normal vector

$$\mathbf{n} = \mp \mathbf{i}b \cos t \pm \mathbf{j}a \sin t.$$

Generally speaking, in this case $\mathbf{n}$ is not a unit vector.

In a neighbourhood of the point t_0 the vector function $\mathbf{r} = \mathbf{r}(t)$ can be expanded by Taylor's formula (or expanded into a vector Taylor series). Let

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j},$$

where $x(t)$ and $y(t)$ have a sufficient number of derivatives in a neighbourhood of the point t_0. Then, expanding these functions by Taylor's formula, we obtain

$$x(t) = x(t_0) + \frac{x'(t_0)}{1!}(t-t_0) + \ldots + \frac{x^{(n)}(t_0)}{n!}(t-t_0)^n + R_n(t), \tag{4}$$

$$y(t) = y(t_0) + \frac{y'(t_0)}{1!}(t-t_0) + \ldots + \frac{y^{(n)}(t_0)}{n!}(t-t_0)^n + \overline{R}_n(t), \tag{5}$$

where $R_n(t)$, $\overline{R_n}(t)$ are remainder terms in some form (Lagrange's, Cauchy's, etc.). Multiplying (4) by $\mathbf{i}$ and (5) by $\mathbf{j}$ and then adding the results, we obtain Taylor's formula for the vector function $\mathbf{r}(t)$

$$\mathbf{r}(t) = \mathbf{r}(t_0) + \frac{\mathbf{r}'(t_0)}{1!}(t-t_0) + \ldots + \frac{\mathbf{r}^{(n)}(t_0)}{n!}(t-t_0)^n + \mathbf{r}_n(t),$$

where the remainder

$$\mathbf{r}_n(t) = R_n(t)\,\mathbf{i} + \overline{R}_n(t)\,\mathbf{j}.$$

Note that if the remainders $R_n(t)$ and $\overline{R}_n(t)$ are written in Lagrange's or Cauchy's form, then the derivatives of order $n+1$ of the functions $x(t)$ and $y(t)$ entering into them are computed, generally speaking, at different points.

CHAPTER 5

INDEFINITE INTEGRALS

Sec. 5.1. Indefinite Integral. Basic Table of Integrals

In the preceding section we introduced the notion of the derivative and learnt to find the derivative of elementary functions. Here we shall solve the inverse problem, namely: given the derivative $f'(x)$ of a function $f(x)$, it is required to find the function $f(x)$ itself.

From the point of view of mechanics, this means that, knowing the velocity of motion of a material point, it is required to find the law of its motion.

Definition. *The function $F(x)$ is called an antiderivative (or primitive) of the function $f(x)$ on an open interval (a, b) if $F(x)$ is differentiable on this interval and $F'(x) = f(x)$.*

An antiderivative of f on a closed interval $[a, b]$ is defined similarly, but in this case one-sided derivatives should be considered at the points a and b (a right derivative at the point a and a left derivative at the point b).

Example 1. $F(x) = \sqrt{x}$ is an antiderivative of the function $f(x) = \frac{1}{2\sqrt{x}}$ on $(0, \infty)$, since $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$.

Example 2. $F(x) = \sin 2x$ is an antiderivative of the function $f(x) = 2\cos 2x$ on $(-\infty, \infty)$, since $(\sin 2x)' = 2\cos 2x$.

Theorem 1. *If $F(x)$ is an antiderivative of the function $f(x)$ on (a, b), then $F(x) + C$ is also an antiderivative, where C is any constant number.*

Proof. We have $(F(x) + C)' = F'(x) = f(x)$.

Theorem 2. *If $F_1(x)$ and $F_2(x)$ are two antiderivatives of $f(x)$ on (a, b), then $F_1(x) - F_2(x) = C$ on (a, b), where C is a certain constant number.*

Proof. By the hypothesis, $F_1'(x) = F_2'(x) = f(x)$. Let us compose the function $\Phi(x) = F_1(x) - F_2(x)$. It is obvious

that

$$\Phi'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0, \quad \forall x \in (a, b).$$

Hence, by the known theorem (see Theorem 6 proved in Sec. 4.12), we conclude that $\Phi(x) \equiv C$, i.e. $F_1(x) - F_2(x) = C$ which was required to be proved.

Thus, it follows from Theorems 1 and 2 that if $F(x)$

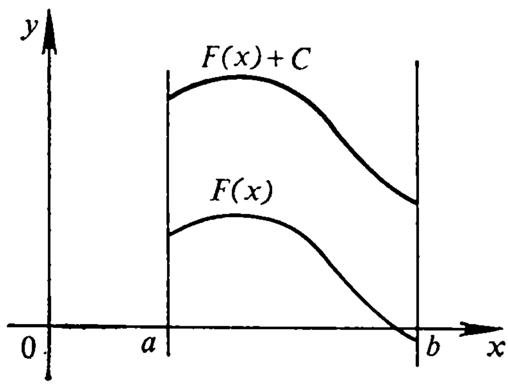

Fig. 73

is an antiderivative of $f(x)$ on (a, b), then any other antiderivative $\Phi(x)$ of $f(x)$ on (a, b) has the form

$$\Phi(x) = F(x) + C, \tag{1}$$

where C is a certain constant number (Fig. 73).

Definition. *An arbitrary antiderivative of $f(x)$ on (a, b) is called the indefinite integral of the function $f(x)$* and is denoted by the symbol

$$\int f(x)\, dx, \tag{2}$$

where $\int$ is the *integral sign*, the expression $f(x)\, dx$ is called the *element of integration*, and $f(x)$ the *integrand*.

If $F(x)$ is one of the antiderivatives of $f(x)$, then, according to the aforesaid,

$$\int f(x)\, dx = F(x) + C, \tag{3}$$

where C is an appropriately chosen constant.

The operation of finding the indefinite integral will be called the *integration* of a function $f(x)$.

Note that if $F(x)$ is an antiderivative of the function $f(x)$, then the element of integration $f(x)\,dx = F'(x)\,dx = dF(x)$ is the differential of the antiderivative $F(x)$.

We shall prove below (see Sec. 6.3) that *if* $f(x)$ *is continuous on* (a, b), *then it has an antiderivative on* (a, b) and, consequently, an indefinite integral.

Let us note a number of properties of the indefinite integral which follow from its definition.

1°. $d\int f(x)\,dx = f(x)\,dx$. Indeed, $\int f(x)\,dx = F(x)+C$,

hence

$$d\int f(x)\,dx = d\,(F(x)+C) = dF(x) = F'(x)\,dx = f(x)\,dx.$$

2°. $\int dF(x) = F(x)+C$, i.e. $\int$ and d are also mutually reduced but we must add to $F(x)$ a certain constant C. We have $\int dF(x) = \int F'(x)\,dx =$ (by definition) $= F(x)+C$.

3°. $\int Af(x)\,dx = A\int f(x)\,dx + C$, where A is a constant number and C is a certain constant.

4°. $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx + C$, where C is a certain constant.

Indeed,

$$\left(\int f(x)\,dx + \int g(x)\,dx\right)' = \left(\int f(x)\,dx\right)' + \left(\int g(x)\,dx\right)'$$
$$= \text{(by definition)} = f(x)+g(x)$$

On the other hand,

$$\left(\int [f(x)+g(x)]\,dx\right)' = \text{(by definition)} = f(x)+g(x).$$

Thus, both the function $\int f\,dx + \int g\,dx$ and the function $\int [f+g]\,dx$ are antiderivatives of one and

the same function $f + g$. But then they differ by a certain constant C which is just written in equality 4°.

5°. If $F(x)$ is an antiderivative of $f(x)$, then

$$\int f(ax+b)\,dx = \frac{1}{a} F(ax+b) + C.$$

Indeed,

$$\left[\frac{1}{a} F(ax+b)\right]' = \frac{1}{a} \cdot aF'(ax+b) = f(ax+b).$$

Let us write down the table of integrals directly implied by the differentiation formulas for the basic elementary functions.

1. $\int 0 \cdot dx = C.$

2. $\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \quad \forall \alpha \neq -1.$

3. $\int x^{-1}\,dx = \int \frac{dx}{x} = \ln|x| + C$, on an interval not containing $x = 0$.

4. $\int a^x\,dx = \frac{a^x}{\ln a} + C \ (0 < a,\ a \neq 1), \int e^x\,dx = e^x + C.$

5. $\int \sin x\,dx = -\cos x + C, \int \cos x\,dx = \sin x + C.$

6. $\int \frac{dx}{\cos^2 x} = \tan x + C, \int \frac{dx}{\sin^2 x} = -\cot x + C$, on an interval where the integrand is continuous.

7. $\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x + C, \\ -\arccos x + C \end{cases} \quad (-1 < x < 1).$

8. $\int \frac{dx}{1+x^2} = \begin{cases} \arctan x + C, \\ -\operatorname{arccot} x + C. \end{cases}$

9. $\int \sinh x\,dx = \cosh x + C, \int \cosh x\,dx = \sinh x + C.$

10. $\int \frac{dx}{\cosh^2 x} = \tanh x + C, \int \frac{dx}{\sinh^2 x}$
$= -\coth x + C \ (x \neq 0).$

11. $\int \frac{dx}{\sqrt{x^2+1}} = \ln|x+\sqrt{x^2+1}|+C = \operatorname{Arsinh} x + C,$

$$\int \frac{dx}{\sqrt{x^2-1}} = \ln|x+\sqrt{x^2-1}|+C = \operatorname{Arcosh} x + C \qquad (|x|>1).$$

12. $\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|+C \quad (|x|\neq 1).$

On the left-hand side of each of these equalities stands an arbitrary (but definitely chosen) antiderivative of the corresponding integrand function, while on the right-hand side there is a concrete antiderivative to which a constant C (we call it an arbitrary constant or a constant of integration) is added such that the equality between these antiderivatives holds.

Let us prove, say, formula 3. Since for $x \neq 0$ we have $|x|' = \operatorname{sgn} x$ and $x \operatorname{sgn} x = |x|$, then

$$(\ln|x|+C)' = \frac{1}{|x|}(|x|)' = \frac{\operatorname{sgn} x}{|x|} = \frac{1}{x} \qquad (x\neq 0),$$

and formula 3 has been proved.

Let us also prove formula 11:

$$(\ln|x+\sqrt{x^2+1}|+C)'$$
$$= \frac{\operatorname{sgn}(x+\sqrt{x^2+1})}{|x+\sqrt{x^2+1}|}\left(1+\frac{x}{\sqrt{x^2+1}}\right)$$
$$= \frac{(x+\sqrt{x^2+1})\operatorname{sgn}(x+\sqrt{x^2+1})}{\sqrt{x^2+1}\,|x+\sqrt{x^2+1}|} = \frac{1}{\sqrt{x^2+1}},$$

and formula 11 has been proved.

On the other hand, $\int \frac{dx}{\sqrt{x^2+1}} = \operatorname{Arsinh} x + C$, therefore, by Theorem 2, $\operatorname{Arsinh} x = \ln|x+\sqrt{x^2+1}|+C$. But since $\operatorname{Arsinh} 0 = 0$, we have $\ln|x+\sqrt{x^2+1}| = \operatorname{Arsinh} x$ (see Sec. 4.6, item 9).

Using property 5°, we can write a more complicated table of integrals. For example:

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+C.$$

Note that the operation of differentiation of elementary functions leads again to elementary functions, whereas the operation of integration can lead to nonelementary functions, i.e. to functions which are not expressible by a finite number of arithmetic operations and superpositions of elementary functions.

For instance, it is proved that the following integrals are nonintegrable in elementary functions:

$\int e^{-x^2}\,dx$ — Poisson's integral,

$\int \cos x^2\,dx$, $\int \sin x^2\,dx$ — Fresnel integrals,

$\int \frac{dx}{\ln x}$ — integral logarithm,

$\int \frac{\cos x}{x}\,dx$ — integral cosine,

$\int \frac{\sin x}{x}\,dx$ — integral sine.

These integrals exist, but cannot be represented in terms of elementary functions. There are other methods for their computation. For instance, the integral sine can be represented in the form of an infinite power series (see Sec. 4.16)

$$\int \frac{\sin x}{x}\,dx = C + x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} + \ldots .$$

Sec. 5.2. Methods of Integration

One of the most important methods of the integral calculus is integration by change of variable (by substitution):

$$\int f(x)\,dx = \int f(\varphi(t))\,\varphi'(t)\,dt + C$$
$$= \int f(\varphi(t))\,d\varphi(t) + C. \qquad (1)$$

In this formula it is supposed that $x = \varphi(t)$ is a continuously differentiable function (i.e. having a continuous derivative) in an interval of variation of t and $f(x)$ is a continuous function in the corresponding interval

of the x-axis. The first equality of (1) asserts that its left-hand side is identically equal to the right-hand side in which, after the integration has been performed, the substitution $x = \varphi(t)$ is made and the constant C is appropriately chosen. Let us prove what has been said.

On the left-hand side of (1) stands a function which is an antiderivative of $f(x)$. If x is replaced by $x = \varphi(t)$ in this antiderivative, then its derivative with respect to t is

$$\frac{d}{dt}\int f(x)\,dx = \frac{d}{dx}\left(\int f(x)\,dx\right)\frac{dx}{dt} = f(\varphi(t))\,\varphi'(t).$$

Consequently, if we make the substitution $x = \varphi(t)$ in the antiderivative of $f(x)$, then the resultant expression is an antiderivative of $f(\varphi(t))\,\varphi'(t)$. As to the integral on the right-hand of (1), it gives, by definition, the general expression for an arbitrary antiderivative of $f(\varphi(t))\,\varphi'(t)$. As we know, two antiderivatives of one and the same function differ from each other only by a constant C. This very fact is expressed by the first equality of (1). The second equality is of a purely formal character: we simply agree to write

$$\int F(t)\,\varphi'(t)\,dt = \int F(t)\,d\varphi(t). \tag{2}$$

For instance,

$$\int e^{x^2} x\,dx = \frac{1}{2}\int e^{x^2}\,2x\,dx + C = \frac{1}{2}\int e^{x^2}\,dx^2 + C$$
$$= \frac{1}{2}\int e^u\,du + C_1 = \frac{1}{2}e^u + C_2 = \frac{1}{2}e^x + C_2 \quad (u = x^2). \tag{3}$$

The first equality in (3) is written by virtue of 3° (Sec. 5.1), the second by virtue of (2), the third equality corresponds to (1) (the constant has been changed here), and the fourth equality is written by virtue of the tabular formula (where the constant has been changed again). In practical computations we usually do not write the arbitrary constant C in the intermediate expressions containing indefinite integrals; for instance, the above calculations (3) can be written in a simplified form

$$\int e^{x^2} x\,dx = \frac{1}{2}\int e^{x^2} 2x\,dx = \frac{1}{2}\int e^{x^2}\,dx^2 = \frac{1}{2}e^{x^2} + C,$$

where the obvious third and fourth equalities are omitted.

Here is another example: $I = \int \sqrt{a^2 - x^2}\, dx$, $a > 0$. The table of integrals does not contain this integral. Setting $x = a \sin t$, we get $\sqrt{a^2 - x^2} = a\sqrt{1 - \sin^2 t} = a \cos t$ and $dx = a \cos t\, dt$. Consequently,

$$I = \int a \cos t\, a \cos t\, dt = a^2 \int \cos^2 t\, dt = a^2 \int \frac{1+\cos 2t}{2}\, dt$$

$$= \frac{a^2 t}{2} + \frac{a^2}{4} \sin 2t + C.$$

But $t = \arcsin \frac{x}{a}$, therefore

$$I = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{a^2}{2} \sin t \cos t + C = \frac{a^2}{2} \arcsin \frac{x}{a}$$

$$+ \frac{a^2}{2} \frac{x}{a} \sqrt{1 - \left(\frac{x}{a}\right)^2} + C = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C.$$

Thus

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C.$$

We also give some more examples which will be of use for the theory of integration of rational fractions:

$$\int \frac{dx}{(x-a)^m} = \int \frac{d(x-a)}{(x-a)^m} = \frac{1}{(x-a)^{m-1}(1-m)} + C \quad (m \neq 1); \tag{4}$$

$$\int \frac{dx}{x-a} = \int \frac{d(x-a)}{x-a} = \ln |x-a| + C; \tag{5}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \int \frac{d(x/a)}{1+(x/a)^2} = \frac{1}{a} \arctan \frac{x}{a} + C;$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \int \left(\frac{1}{x-a} - \frac{1}{x+a}\right) dx$$

$$= \frac{1}{2a} (\ln |x-a| - \ln |x+a|) + C = \frac{1}{2a} \ln \left|\frac{x-a}{x+a}\right| + C;$$

$$\int \frac{dx}{x^2+px+q} = \int \frac{dx}{(x+(p/2))^2}$$

$$= \int \frac{d(x+(p/2))}{(x+(p/2))^2} = -\frac{1}{x+(p/2)} + C \left(q - \frac{p^2}{4} = 0\right);$$

$$\int \frac{dx}{x^2+px+q} = \int \frac{dx}{(x+(p/2))^2+(q-(p^2/4))}$$

$$= \int \frac{d(x+(p/2))}{(x+(p/2))^2+a^2} = \frac{1}{a} \int \frac{d\left(\frac{x+(p/2)}{a}\right)}{1+\left(\frac{x+(p/2)}{a}\right)^2}$$

$$= \frac{1}{a} \operatorname{arc\,tan} \frac{x+(p/2)}{a} + C \left(q - \frac{p^2}{4} = a^2,\ a > 0\right); \qquad (6)$$

$$\int \frac{dx}{x^2+px+q} = \int \frac{d(x+(p/2))}{(x+(p/2))^2-a^2}$$

$$= \frac{1}{2a} \ln \left| \frac{x+(p/2)-a}{x+(p/2)+a} \right| + C \quad \left(q - \frac{p^2}{4} = -a^2,\ a > 0\right);$$

$$\int \frac{(2x+p)\,dx}{x^2+px+q} = \int \frac{d(x^2+px+q)}{x^2+px+q} = \ln |x^2+px+q| + C;$$

$$\int \frac{Ax+B}{x^2+px+q}\,dx = \frac{A}{2} \int \frac{2x+(2B/A)}{x^2+px+q}\,dx = \frac{A}{2} \int \frac{(2x+p)\,dx}{x^2+px+q}$$

$$+ \frac{A}{2} \int \frac{(2B/A)-p}{x^2+px+q}\,dx = \frac{A}{2} \ln |x^2+px+q|$$

$$+ D \int \frac{dx}{x^2+px+q}, \qquad (7)$$

$$\left(A \neq 0,\ D = \frac{A}{2}\left(\frac{2B}{A} - p\right)\right) \quad \text{(see (6))}.$$

For the theory of integration of rational fractions it is important that the computation of integrals (4)-(7), in which a, A, B, p, and q are constants, leads to elementary functions (namely, to rational functions, logarithmic functions, and the arc tangent).

Now we proceed to the *formula of integration by parts*:

$$\int uv'\,dx = uv - \int vu'\,dx + C \qquad (8)$$

or, which is the same,

$$\int u\,dv = uv - \int v\,du + C.$$

Since the right-hand member of (8) contains an indefinite integral, the constant C is usually omitted.

It is assumed in this formula that $u(x)$ and $v(x)$ are continuously differentiable functions. The validity of

formula (8) follows from the fact that the derivatives of the left-hand and right-hand members are equal:

$$uv' = (uv)' - vu'.$$

Formula (8) reduces the computation of the integral $\int u\,dv$ to evaluation of the integral $\int v\,du$. Computation by formula (8) is called the *method of integration by parts.*

Example 1. Compute $\int x \ln x\,dx$. Let us set

$$\begin{array}{l|l} u(x) = \ln x, & du = \dfrac{dx}{x}, \\ x\,dx = dv, & v = \displaystyle\int dv = \int x\,dx = \frac{x^2}{2}. \end{array}$$

Then

$$\int x \ln x\,dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{dx}{x} = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C.$$

Example 2. Compute the integrals $I = \int e^{ax} \sin bx\,dx$, $I_1 = \int e^{ax} \cos bx\,dx$, where a and b are constant numbers. In this case, the element of integration can be represented in the form of a product of $u(x)$ and $dv(x)$ in two ways: $u = e^{ax}$, $dv = \sin bx\,dx$ or $u = \sin bx$, $dv = e^{ax}\,dx$.

And so, let

$$\begin{array}{r|l} u = e^{ax}, & du = ae^{ax}\,dx, \\ \sin bx\,dx = dv, & v = -\dfrac{\cos bx}{b}. \end{array}$$

Then, applying the rule for integration by parts, we have

$$I = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx\,dx$$
$$= -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} I_1. \qquad (9)$$

To the integral I_1 we also apply the method of integration by parts, setting $u = e^{ax}$, $dv = \cos bx\,dx$. Then

$$I_1 = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} I. \qquad (10)$$

From (9) and (10) we obtain a system for determining I and I_1:

$$\left.\begin{aligned} I-\frac{a}{b}I_1&=-\frac{1}{b}e^{ax}\cos bx,\\ \frac{a}{b}I+I_1&=\frac{1}{b}e^{ax}\sin bx. \end{aligned}\right\}$$

Solving this system, we get

$$I=\frac{a\sin bx-b\cos bx}{a^2+b^2}e^{ax}+C,\quad I_1=\frac{b\sin bx+a\cos bx}{a^2+b^2}e^{ax}+C.$$

Example 3. Compute the integral $I=\int \operatorname{arc\,sin} x\,dx$.

Setting $u=\operatorname{arc\,sin} x$, $dv=dx$, we obtain

$$I=x\operatorname{arc\,sin} x-\int\frac{x\,dx}{\sqrt{1-x^2}}=x\operatorname{arc\,sin} x+\sqrt{1-x^2}+C.$$

Example 4. Let us give one more example which will be of use in the theory of integration of rational fractions. Let $k>1$ be a natural number and let $a>0$; then

$$\int\frac{dx}{(x^2+a^2)^{k-1}}=a^2\int\frac{dx}{(x^2+a^2)^k}+\frac{1}{2}\int\frac{x\cdot 2x\,dx}{(x^2+a^2)^k}$$

$$=\left(u=x,\ dv=\frac{2x\,dx}{(x^2+a^2)^k}\right)=a^2\int\frac{dx}{(x^2+a^2)^k}$$

$$+\frac{1}{2}\left\{\frac{x}{(1-k)(x^2+a^2)^{k-1}}-\frac{1}{1-k}\int\frac{dx}{(x^2+a^2)^{k-1}}\right\},$$

whence

$$a^2\int\frac{dx}{(x^2+a^2)^k}=\frac{x}{2(k-1)(x^2+a^2)^{k-1}}+\frac{2k-3}{2(k-1)}\int\frac{dx}{(x^2+a^2)^{k-1}}.$$

Now, if $k>2$, the integral on the right-hand side admits of the application of the same technique, which reduces by unity the exponent in the power function entering as the denominator in the integrand fraction. Proceeding in this way, we ultimately arrive at an integral of $(x^2+a^2)^{-1}$ which is reducible to the arc tangent.

Thus, for $q-(p^2/4)=a^2>0$ and any natural k the integral

$$\int\frac{dx}{(x^2+px+q)^k}=\int\frac{du}{(u^2+a^2)^k}+C\left(u=x+\frac{p}{2}\right)\tag{11}$$

is expressible in terms of elementary functions.

Example 5. Compute the integrals

$$\int P_n(x)\begin{Bmatrix} e^{bx} \\ \cos bx \\ \sin bx \end{Bmatrix} dx,$$

where $P_n(x) = a_n x^n + \ldots + a_1 x + a_0$ is an algebraic polynomial of degree n.

The given integrals are computed by n-fold application of the method of integration by parts, consecutively setting $u = P_n(x)$, then $u = P_n'(x)$, ... The integrals thus obtained will be simplified, since the derivative of the algebraic polynomial $P_n(x)$ is again an algebraic polynomial of degree reduced by unity.

Since the character of the antiderivative of the functions under consideration is easily guessed, these integrals can be computed using the so-called *method of undetermined coefficients.*

For instance, for $\int P_n(x) e^{bx} dx$ the antiderivative has the form $Q_n(x) e^{bx} + C$, where $Q_n(x) = b_n x^n + \ldots + b_1 x + b_0$ and $b_0, \ldots, b_n$ are, for the time being, undetermined coefficients. We find these coefficients from the conditions that

$$(Q_n(x) e^{bx} + C)' = P_n(x) e^{bx} \quad \text{or}$$

$$Q_n'(x) + bQ_n(x) = P_n(x).$$

Equating the coefficients of the identical powers of x, we just find all the numbers $b_0, \ldots, b_n$. This method is called the *method of undetermined coefficients.* Here we have taken an advantage of the fact that two polynomials are equal to each other if and only if the coefficients of the corresponding powers of x are equal (see Sec. 4.14, Theorem 2).

Let us illustrate this by a particular example:

$$\int (x^2+1) e^x dx = (ax^2+bx+c) e^x + C.$$

In this case

$$P_2(x) = x^2 + 1, \quad Q_2(x) = ax^2 + bx + c,$$

where the coefficients a, b, c should be determined. We have

$$(Q_2(x)\,e^x)' = [ax^2 + (2a + b)\,x + b + c]\,e^x$$
$$= (x^2 + 1)\,e^x,$$

whence $ax^2 + (2a + b)\,x + b + c = x^2 + 1$. Since this equality must hold for all x's, the coefficients of the identical powers of x in its left and right members are equal to each other (Sec. 4.14, (15)): $a = 1$, $2a + b = 0$, $b + c = 1$. Hence,

$$\int (x^2 + 1)\,e^x dx = (x^2 - 2x + 3)\,e^x + C.$$

Sec. 5.3. Complex Numbers

Complex numbers are defined as the expressions

$$z = a + bi = a + ib,$$

where a, b are real numbers, and i is a special symbol; in addition, for the complex numbers $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$ the notions of equality and arithmetic operations are introduced according to the following rules:

(1) $z_1 = z_2$ if and only if $a_1 = a_2$ and $b_1 = b_2$; $a + 0i = a$, $0 + bi = bi$, $1 \cdot i = i$.

(2) $z_1 \pm z_2 = (a_1 \pm a_2) + i\,(b_1 \pm b_2)$.

(3) $z_1 \cdot z_2 = (a_1a_2 - b_1b_2) + i\,(b_1a_2 + a_1b_2)$.

(4) $\dfrac{z_1}{z_2} = \dfrac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\,\dfrac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2}$ $(a_2^2 + b_2^2 \neq 0)$.

It follows from (1) and (3) that

$$i^2 = -1.$$

Thus, the above introduced operations of addition and multiplication possess the properties of commutativity ($z_1 + z_2 = z_2 + z_1$, $z_1z_2 = z_2z_1$), associativity ($(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, $(z_1z_2)\,z_3 = z_1\,(z_2z_3)$), and distributivity ($(z_1 + z_2)\,z_3 = z_1z_2 + z_2z_3$).

It should be also noted that we may operate with complex numbers exactly in the same way as we are

used to operate with literal expressions in algebra, the operations being simplified by the fact that $i^2 = -1$.

It follows from the property $a + 0i = a$ that the set of complex numbers contains (as a constituent part) the set of all real numbers. And it is easily seen here that the application of arithmetical operations (2), (3), (4) to the expressions $z_1 = a_1 + 0i$, $z_2 = a_2 + 0i$ leads, respectively, to $a_1 \pm a_2 + 0i = a_1 \pm a_2$, $a_1a_2 + 0i = a_1a_2$, $\frac{a_1}{a_2} + 0i = \frac{a_1}{a_2}$ $(a_2 \neq 0)$.

Given a complex number $z = a + ib$, the number $\bar{z} = a - ib$ is called its *conjugate* (*complex*) *number*. The real number $|z| = \sqrt{a^2 + b^2}$ is called the *modulus* (also the *absolute number*) *of the complex number* z. It is obvious that $z \cdot \bar{z} = |z|^2$.

If a complex number $z = a + ib$ is treated as a point (vector) $M(a, b)$ of the plane xOy, then $|z|$ is equal to the

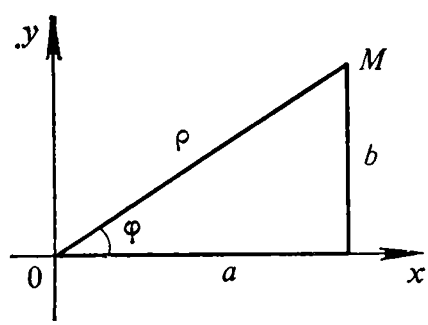

Fig. 74

distance of the point $M(a, b)$ from the origin (Fig. 74).

If the polar coordinates (ρ, φ) are introduced in the plane, then

$$\begin{aligned} a &= \rho \cos \varphi = |z| \cos \varphi, \\ b &= \rho \sin \varphi = |z| \sin \varphi \end{aligned} \qquad (|z| > 0). \tag{1}$$

By virtue of (1), the complex number z can be written in the form

$$z = \rho(\cos \varphi + i \sin \varphi), \tag{2}$$

where ρ is the modulus of the number z, and φ is the angle (in radians) formed by the vector $\overrightarrow{OM}$ with the positive direction of the x-axis. This angle is also called the *argument of the complex number* z and is denoted $\varphi = \arg z$ $(0 \leqslant \varphi < 2\pi)$.

Obviously, $\varphi = \arg z$ is a one-valued function of $z \neq 0$. A many-valued or rather infinite-valued function (the *argument* z with a capital letter) is also introduced

$$\varphi = \operatorname{Arg} z = \arg z + 2k\pi \quad (k = 0, \pm 1, \pm 2, \ldots).$$

The latter function gives all the values of φ for which, for the given $z \neq 0$, two equalities (1) are satisfied.

Note that $z = 0$ is the only number for which its argument makes no sense, but, in return, it can be defined as the number whose modulus is equal to zero ($|z| = 0$).

Written with a small letter, $\arg z$ is also called the *argument in the reduced form*. It is sometimes convenient to regard the angle belonging to the other half-interval $[a, a + 2\pi)$ of length 2π, say to $[-\pi, \pi)$ as the argument in the reduced form.

The numbers a and b are called the *real* and the *imaginary* parts, respectively, and are denoted: $a = \operatorname{Re} z$, $b = \operatorname{Im} z$. Thus,

$$z = \operatorname{Re} z + i \operatorname{Im} z.$$

By definition,

$$e^{i\varphi} = \cos\varphi + i \sin\varphi \quad (-\infty < \varphi < \infty). \tag{3}$$

It is obvious that $e^{i\varphi}$ is a complex function (attaining complex values) of the real argument φ. It is clear that $e^{i\varphi}$ is a periodic function of period 2π: $e^{i(\varphi+2\pi)} = e^{i\varphi}$.

Since $|e^{i\varphi}| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$, the point $e^{i\varphi}$ describes continuously a circle of radius 1 with centre at the point $z = 0$, when φ varies continuously in the half-interval $0 \leqslant \varphi < 2\pi$.

The following equalities hold true:

$$e^{i(\varphi_1+\varphi_2)} = e^{i\varphi_1} e^{i\varphi_2}, \quad e^{-i\varphi} = \frac{1}{e^{i\varphi}}. \tag{4}$$

Indeed,

$$e^{i(\varphi_1+\varphi_2)} = \cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)$$
$$= (\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2)$$
$$+ i(\sin\varphi_1\cos\varphi_2 + \sin\varphi_2\cos\varphi_1)$$
$$= (\cos\varphi_1 + i\sin\varphi_1)(\cos\varphi_2 + i\sin\varphi_1) = e^{i\varphi_1}e^{i\varphi_2},$$

$$\frac{1}{e^{i\varphi}} = \frac{1}{\cos\varphi + i\sin\varphi} = \cos\varphi - i\sin\varphi$$
$$= \cos(-\varphi) + i\sin(-\varphi) = e^{-i\varphi}$$

For an arbitrary complex variable $z = x + iy$ the function e^z is defined with the aid of the equality

$$e^z = e^x e^{iy}, \quad z \neq 0.$$

Hence, by (3),

$$e^z = e^x(\cos y + i\sin y). \tag{5}$$

On the basis of (2), (3), any complex number z can be represented in the form

$$z = \rho e^{i\varphi} \ (\rho \geqslant 0), \tag{6}$$

where the nonnegative number $\rho = |z|$ is unique for the given z and for $\rho > 0$ the angle

$$\varphi = \operatorname{Arg} z = \arg z + 2k\pi$$

is determined with an accuracy up to $2k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$).

Expressions (2) and (6) are called the *trigonometric and the exponential forms of a complex number* z, respectively.

Given below are some examples of complex numbers written in the exponential form (assuming $\varphi = \arg z$):

$$1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}e^{i\pi/4},$$

$$i = 0 + 1\cdot i = e^{i\pi/2}, \quad 1 = e^{0i}, \quad -1 = e^{\pi i}.$$

From equalities (3), (4) we easily obtain De Moivre's formula

$$(\cos\varphi + i\sin\varphi)^n = e^{in\varphi} = \cos n\varphi + i\sin n\varphi. \tag{7}$$

The following equality is also valid:

$$z_1 z_2 = |z_1|\,|z_2|\, e^{i(\varphi_1+\varphi_2)},$$

i.e. as complex numbers are multiplied, their moduli are multiplied and the arguments are added irrespective of the form they are taken in (reduced or not).

The operation of constructing a conjugate complex number possesses the following simple properties:

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}\ (z_2 \neq 0). \qquad (8)$$

Indeed,

$$\overline{(a_1+b_1 i) \pm (a_2+b_2 i)} = \overline{(a_1 \pm a_2) + (b_1 \pm b_2)\, i}$$
$$= (a_1 \pm a_2) - i\,(b_1 \pm b_2) = (a_1 - b_1 i) \pm (a_2 - b_2 i)$$
$$= \overline{(a_1+b_1 i)} \pm \overline{(a_2+b_2 i)};$$

further, since

$$\overline{\rho e^{i\varphi}} = \overline{\rho(\cos\varphi + i\sin\varphi)} = \rho(\cos\varphi - i\sin\varphi) = \rho e^{-i\varphi},$$

we have

$$\overline{z_1 z_2} = \overline{\rho_1 e^{i\varphi_1}\rho_2 e^{i\varphi_2}} = \overline{\rho_1\rho_2 e^{i(\varphi_1+\varphi_2)}} = \rho_1\rho_2 e^{-i(\varphi_1+\varphi_2)}$$
$$= \rho_1 e^{-i\varphi_1}\rho_2 e^{-i\varphi_2} = \overline{\rho_1 e^{i\varphi_1}} \cdot \overline{\rho_2 e^{i\varphi_2}} = \bar{z}_1 \cdot \bar{z}_2.$$

In the case of a quotient, a similar proof takes place.

Let us consider the problem of computing the nth root of a complex number $a = \rho e^{i\theta}$ ($\rho > 0$). It is required to find all the numbers $b = re^{i\varphi}$ such that $b^n = a$. For such b's we have $r^n e^{in\varphi} = \rho e^{i\theta}$ (r, $\rho > 0$) and, by the uniqueness of representation of a complex number in the exponential form, we obtain $\rho = r^n$, $n\varphi = \theta + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$. From the first of these equalities it follows that $r = \sqrt[n]{\rho}$ (r is the arithmetic nth root of the positive number ρ). The second of these equalities shows that $\varphi = \frac{\theta}{n} + \frac{2k\pi}{n}$ ($k = 0, \pm 1, \ldots$).

The values of φ which yield essentially different values of the nth root of a correspond only to n successive values of k, for instance,

$$\varphi_k = \frac{\theta}{n} + \frac{2k\pi}{n}\ (k = 0, 1, \ldots, n-1). \qquad (9)$$

To the other integral values of k there correspond the values of φ differing from one of the values (9) by a quantity multiple of 2π.

We have proved that there are exactly n different values of the nth root of a complex number $a \neq 0$ expressed by the formula

$$\sqrt[n]{a}=\sqrt[n]{\rho e^{i\theta}}=\sqrt[n]{\rho}\, e^{i\varphi_k} \quad (k=0, 1, \ldots, n-1),$$

where φ_k is given by equalities (9).

Examples:

(1) $\sqrt[3]{1}=\sqrt[3]{e^{0i}}=e^{\left(\frac{0}{3}+\frac{2k\pi}{3}\right)i} \quad (k=0, 1, 2).$

(2) $\sqrt[3]{i}=\sqrt[3]{e^{i\frac{\pi}{2}}}=e^{\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right)i} \quad (k=0, 1, 2).$

(3) $\sqrt[6]{1+i}=\sqrt[12]{2}\sqrt[6]{e^{i\frac{\pi}{4}}}=\sqrt[12]{2}e^{i\left(\frac{\pi}{24}+\frac{2k\pi}{6}\right)}$

$(k=0, 1, \ldots, 5).$

Sec. 5.4. Theory of Polynomial of *n*th Degree

The polynomial of the nth degree is defined as a function of the form

$$Q_n(z)=a_0+a_1z+\ldots+a_nz^n=\sum_{k=0}^{n} a_kz^k, \tag{1}$$

where a_k are constant coefficients, real or complex, and z is a variable, generally speaking, a complex one, assuming any complex values ($z = x+iy$), or, speaking geometrically, z can represent any point of the complex plane.

With the aid of formula (1), every point z of the complex plane is associated with the number $Q_n(z)$, which is, generally speaking, a complex one. Henceforward, we shall assume that $a_n \neq 0$. If $Q_n(a) = 0$, then the number a is called the *root* or *zero* of the polynomial $Q_n(z)$.

Reasoning exactly in the same way as at the beginning of Sec. 4.14, where we dealt with the polynomial of a real variable, we can show that for any complex number z_0

the polynomial $Q_n(z)$ is expanded in powers of $z - z_0$ in a unique way, i.e. it is represented in the form

$$Q_n(z) = \sum_{k=0}^{n} b_k (z - z_0)^k,$$

where b_k are constant numbers, generally speaking, complex numbers. Obviously, $Q_n(z_0) = b_0$. Hence it follows that in order for a point z_0 to be a root of the polynomial Q_n, it is necessary and sufficient that the zero coefficient b_0 of the expansion of Q_n in powers of $z - z_0$ be equal to zero ($b_0 = 0$). But if $b_0 = 0$, then Q_n can be represented in the form

$$Q_n(z) = (z - z_0) Q_{n-1}(z), \quad \forall z, \tag{2}$$

where Q_{n-1} is a certain polynomial of degree $n - 1$. Conversely, if Q_n can be represented in form (2), or, in other words, if $Q_n(z)$ is divisible by $z - z_0$ without a remainder, then, obviously, z_0 is a root of Q_n.

Thus, we have proved the following theorem:

Bezout's Theorem. *For a polynomial $Q_n(z)$ to have a (complex) root z_0, it is necessary and sufficient that it be divisible by $z - z_0$, i.e. be representable in the form of product* (2), *where Q_{n-1} is a certain polynomial of degree $n - 1$.*

Let z_0 be a root of Q_n and, thus, representation (2) takes place. If in this case $Q_{n-1}(z_0) \neq 0$, then by Bezout's theorem applied to Q_{n-1}, the polynomial $Q_{n-1}(z)$ is not divisible by $z - z_0$, and $Q_n(z)$ although is divisible by $z - z_0$, but is not divisible by $(z - z_0)^2$. In this case z_0 is said to be a *simple root* (*zero*) of the polynomial Q_n. Let now $Q_{n-1}(z_0) = 0$; then, by Bezout's theorem applied to $Q_{n-1}(z)$, the polynomial $Q_{n-1}(z)$ is divisible by $z - z_0$ and we shall obtain the equality $Q_n(z) = (z - z_0)^2 \times \times Q_{n-2}(z)$, where $Q_{n-2}(z)$ is a certain polynomial of degree $n - 2$. If $Q_{n-2}(z_0) \neq 0$, then $Q_n(z)$ is divisible by $(z - z_0)^2$, but is not divisible by $(z - z_0)^3$, and then the number z_0 is called a root (zero) of multiplicity 2. In the general case for some natural $s \leqslant n$ we have

$$Q_n(z) = (z - z_0)^s Q_{n-s}(z), \quad Q_{n-s}(z_0) \neq 0,$$

where $Q_{n-s}(z)$ is a polynomial of degree $n-s$, and then z_0 is said to be a *root (zero) of a polynomial Q_n of multiplicity s.*

The following theorem holds true.

Basic Theorem. *Any polynomial of degree n has at least one complex root (zero).*

We are not going to prove this theorem here. This fundamental theorem of algebra only guarantees the existence of a root (which is complex in the general case) of a polynomial of the nth degree, but does not provide any effective means for its computation. It should be mentioned that the proof of this theorem is carried out with the aid of mathematical analysis, not algebraically, and the only reason why this proof is not given here is that it does in fact have a closer relation to the theory of functions of a complex variable which we do not treat in this book.

An important corollary follows from the above theorem.

Corollary. *A polynomial of the nth degree Q_n with a leading nonzero coefficient ($a_n \neq 0$) has n complex roots with regard to multiplicity, in other words, $Q_n(z)$ is representable in the form of a product:*

$$Q_n(z) = a_n (z-z_1)^{p_1} (z-z_2)^{p_2} \dots (z-z_l)^{p_l}, \qquad (3)$$
$$p_1 + p_2 + \dots + p_l = n,$$

where $z_1, \dots, z_l$ are different roots of Q_n of respective multiplicities $p_1, \dots, p_l$.

Proof. According to the fundamental theorem, the polynomial Q_n has at least one root. Let us denote it by z_1 and its multiplicity by p_1. Then

$$Q_n(z) = (z-z_1)^{p_1} Q_{n-p_1}(z), \quad Q_{n-p_1}(z_1) \neq 0.$$

If $n-p_1 = 0$, i.e. $p_1 = n$, then, necessarily, $Q_{n-p_1}(z) = a_n$ and the statement has been proved. In this case $Q_n(z) = a_n(z-z_1)^n$.

And if $p_1 < n$, then $Q_{n-p_1}(z)$ is a polynomial of degree $n-p_1$, not divisible by $z-z_1$, and its leading coefficient is not equal to zero. To this polynomial we can apply the fundamental theorem, by which it has a complex root. Denoting the latter by z_2 and its multiplicity by p_2, we obtain

$$Q_n(z) = (z-z_1)^{p_1} (z-z_2)^{p_2} Q_{n-p_1-p_2}(z)$$
$$(Q_{n-p_1-p_2}(z_j) \neq 0, \quad j = 1, 2).$$

If $n - p_1 - p_2 = 0$, then $Q_{n-p_1-p_2}(z) = a_n$. If not, then the process can be continued. But after a finite number of such steps (not exceeding n), this process comes to an end and we obtain formula (3). If z is replaced by a number different from $z_1, \ldots, z_l$ in the right-hand side of (3), then the formula does not vanish. This indicates that the polynomial Q_n has no other roots (except for the found ones) and representation (3) is unique.

Sec. 5.5. Real Polynomial of *n*th Degree

A polynomial

$$Q_n(z) = \sum_{k=0}^{n} a_k z^k \quad (a_n \neq 0) \tag{1}$$

is called *real* if all its coefficients a_k are real numbers. It is termed so due to the fact that a real polynomial, when considered only for a real variable $z = x$, is a real function $Q(x)$ of the real variable x, that is a function assuming real values. Of course, for complex z's a real polynomial assumes, generally speaking, complex values.

Lemma. *For a real polynomial $Q_n(z)$ there holds the equality*

$$Q_n(\bar{z}) = \overline{Q_n(z)}, \quad \forall z.$$

Proof. This relation is derived on the basis of equalities (8) from Sec. 5.3 and the fact that for real a_k we have $a_k = \bar{a}_k$.

We have

$$Q_n(\bar{z}) = \sum_{k=0}^{n} a_k \bar{z}^k = \sum_{k=0}^{n} \bar{a}_k \overline{z^k} = \sum_{k=0}^{n} \overline{a_k z^k} = \overline{\sum_{k=0}^{n} a_k z^k}$$

$$= \overline{Q_n(z)}, \tag{2}$$

which was required to be proved.

Theorem 1. *If a real polynomial Q_n has a complex root $z_0 = \alpha + i\beta$ $(\beta \neq 0)$ of multiplicity ν, then it also has the conjugate complex root $\bar{z}_0 = \alpha - i\beta$ of the same multiplicity, and then*

$$Q_n(z) = [(z-\alpha)^2 + \beta^2]^\nu \, Q_{n-2\nu}(z), \tag{3}$$

where $Q_{n-2\nu}(z)$ is a real polynomial of degree $n - 2\nu$ which is not equal to zero for $z = z_0$ and $z = \bar{z}_0$.

Proof. Let $z_0 = \alpha + i\beta$ ($\beta \neq 0$) be a root of Q_n. Then $\bar{z}_0 = \alpha - i\beta$ is also a root of Q_n, since, by virtue of (2), $Q_n(\bar{z}_0) = \overline{Q_n(z_0)} = \bar{0} = 0$. The numbers z_0 and $\bar{z}_0$ are not equal to each other and $Q_n(z)$ is divisible by

$$(z - \alpha - i\beta)(z - \alpha + i\beta) = (z - \alpha)^2 + \beta^2, \quad (4)$$

i.e. by a real polynomial of the second degree. Thus,

$$Q_n(z) = [(z - \alpha)^2 + \beta^2]\, Q_{n-2}(z),$$

where $Q_{n-2}(z)$ is a polynomial of degree $n - 2$, which is, obviously, real, since the quotient, being the result of division of a real polynomial by a real polynomial, is a real polynomial.

If z_0 is a root of Q_n of multiplicity ν and $\nu > 1$, then z_0 is a root of Q_{n-2} of multiplicity $\nu - 1$, therefore, repeating our reasons with respect to $Q_{n-2}(z)$, we conclude that (4) can be factored out of it. Then the second factor will be a real polynomial Q_{n-4} of degree $n - 4$. Repeating this process ν times, we shall obtain the representation of $Q_n(z)$ in form (3), where $Q_{n-2\nu}(z)$ is a real polynomial of degree $n - 2\nu$, possessing the property $Q_{n-2\nu}(z_0) \neq 0$. But then also $Q_{n-2\nu}(\bar{z}_0) \neq 0$. But if $\bar{z}_0$ were a root of the real polynomial $Q_{n-2\nu}$, then z_0 would, inevitably, be a root of this polynomial as well.

Theorem 2. *A real polynomial $Q_n(z)$ with a leading coefficient $a_n \neq 0$ can be represented in the form of the product*

$$Q_n(z) = a_n (z - c_1)^{\mu_1} \ldots (z - c_r)^{\mu_r} [(z - \alpha_1)^2 + \beta_1^2]^{\nu_1}$$

$$\ldots [(z - \alpha_s)^2 + \beta_s^2]^{\nu_s} = a_n \prod_{j=1}^{r} (z - c_j)^{\mu_j} \prod_{j=1}^{s} [(z - \alpha_j)^2 + \beta_j^2]^{\nu_j}, \quad (5)$$

where $\beta_j > 0$, $\mu_1 + \ldots + \mu_r + 2(\nu_1 + \ldots + \nu_s) = n$, $c_1, \ldots, c_r$ are real roots of Q_n of multiplicities $\mu_1, \ldots, \mu_r$, respectively, and $\alpha_1 \pm \beta_1 i, \ldots, \alpha_1 \pm \beta_s i$ are pairwise conjugate complex roots of Q_n of multiplicities $\nu_1, \ldots, \nu_s$.

Remark. The real polynomials of the second degree entering into product (5) can be transformed in the following

ing way:

$$(z-\alpha_j)^2+\beta_j^2 = z^2-2\alpha_j z+(\alpha_j^2+\beta_j^2) = z^2+p_j z+q_j,$$
$$p_j=-2\alpha_j, \quad q_j=\alpha_j^2+\beta_j^2.$$

Therefore formula (5) can also be written in the following form:

$$Q_n(z)=a_n\prod_{j=1}^{r}(z-c_j)^{\mu_j}\prod_{j=1}^{s}(z^2+p_j z+q_j)^{\nu_j}, \qquad (5')$$

where $z^2+p_j z+q_j$ are real polynomials of the second degree having the complex roots $\alpha_j \pm i\beta_j$ ($\beta_j>0$, $p_j^2-4q_j=-4\beta_j^2<0$).

Proof. By formula (3) in Sec. 5.4, we have

$$Q_n(z)=\prod_{j=1}^{r}(z-c_j)^{\mu_j}\,Q_m(z),$$

where $Q_m(z)$ is a real polynomial of degree $m=n-\mu_1-\ldots-\mu_r$. If $m=0$, then, obviously, $Q_m(z)=a_n$; in the general case we apply in succession Theorem 1 to the complex roots of Q_m.

Note that the fundamental theorem only proves the existence of a root (complex, in general) of a polynomial of the nth degree, but does not provide any effective methods for its computation in the general case. However, the proof of this theorem is carried out using the methods of mathematical analysis but not algebra. And the only reason why this proof is not given here is that it does in fact have a closer relation to the theory of functions of a complex variable (the latter being treated in our book *Differential Equations. Multiple Integrals. Series. Functions of a Complex Variable*).

There are formulas for solving general algebraic equations of the second, third, and fourth degrees. As was proved by Abel*, there cannot exist such formulas for equations of degree $n>4$. This should be understood in the sense that for $n>4$ the roots of an equation $a_nx^n+\ldots+a_1x+a_0=0$ ($a_n\neq 0$) cannot be ex-

* Abel, Niels Henrik (1802-1829). Distinguished Norwegian mathematician.

pressed in terms of the coefficients a_k with the aid of functions of these coefficients reducing to a finite number of operations belonging to the following group: addition, subtraction, multiplication, division, and extraction of a root.

Sec. 5.6. Integrating Rational Expressions

The ratio of two algebraic polynomials

$$f(x) = \frac{P_m(x)}{Q_n(x)}, \tag{1}$$

$$P_m(x) = b_0 + b_1 x + \ldots + b_m x^m,$$

$$Q_n(x) = a_0 + a_1 x + \ldots + a_n x^n,$$

b_m, $a_n \neq 0$, $m \geqslant 0$, $n \geqslant 1$, is called a *rational function* or also a *rational fraction.*

We shall hold that the *rational fraction f is real,* i.e. that P_m and Q_n are real polynomials. Besides, we shall assume that x is a real variable.

Rational functions of the form

$$\left.\begin{array}{c} \dfrac{A}{x-a}, \quad \dfrac{A}{(x-a)^k} \ (k \geqslant 2), \quad \dfrac{Ax+B}{x^2+px+q}, \\ \dfrac{Ax+B}{(x^2+px+q)^k} \ (k \geqslant 2) \end{array}\right\} \tag{2}$$

where A, B, a, p, q are real numbers, k is a natural number, and the trinomial $x^2 + px + q$ has no real roots, will be called *partial fractions.*

It was shown in Sec. 5.2 how to compute integrals of partial fractions (see (4), (5), (6), (7), (11) in this section).

Let it be required to find the integral of a rational function $f(x)$ (see (1)). If $m > n$, then, by simple division, we single out the integral part of f:

$$f(x) = \text{polynomial} + \frac{P_{m_1}(x)}{Q_n(x)} \quad (m_1 < n).$$

Integration of a polynomial is rather easy and the difficulty is reduced to integration of a rational fraction in which the numerator is of lower degree than the denominator.

Therefore we shall suppose that our rational fraction $f(x)$ is *proper*, i.e. the degree of its numerator is less than the degree of the denominator ($m < n$).

Theorem 1. *Let the denominator of a proper real rational fraction be factored by formula* (5′) *from the preceding section:*

$$Q_n(x) = a_n (x-c_1)^{\mu_1} \ldots (x-c_r)^{\mu_r} (x^2+p_1x+q_1)^{\nu_1} \ldots$$
$$\ldots (x^2+p_sx+q_s)^{\nu_s}.$$

Then fraction (1) *can be represented in a unique way in the form of the following sum of partial fractions:*

$$\frac{P_m(x)}{Q_n(x)} = \frac{A_{1,1}}{(x-c_1)^{\mu_1}} + \frac{A_{1,2}}{(x-c_1)^{\mu_1-1}} + \ldots + \frac{A_{1,\mu_1}}{x-c_1}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \frac{A_{r,1}}{(x-c_r)^{\mu_r}} + \frac{A_{r,2}}{(x-c_r)^{\mu_r-1}} + \ldots + \frac{A_{r,\mu_r}}{x-c_r}$$
$$+ \frac{B_{1,1}x+C_{1,1}}{(x^2+p_1x+q_1)^{\nu_1}} + \frac{B_{1,2}x+C_{1,2}}{(x^2+p_1x+q_1)^{\nu_1-1}} + \ldots + \frac{B_{1,\nu_1}x+C_{1,\nu_1}}{x^2+p_1x+q_1}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \frac{B_{s,1}x+C_{s,1}}{(x^2+p_sx+q_s)^{\nu_s}} + \frac{B_{s,2}x+C_{s,2}}{(x^2+p_sx+q_s)^{\nu_s-1}} + \ldots + \frac{B_{s,\nu_s}x+C_{s,\nu_s}}{x^2+p_sx+q_s}, \tag{3}$$

where A, B, C (with appropriate indices) are constant numbers.

This theorem asserts that for any proper rational real fraction there exist constant numbers A, B, C with the indicated indices so that identity (3) holds for all x's excepting the values $x = c_1, \ldots, c_r$ for which both sides of (3) are not defined. This theorem can be accurately proved, but we are not going to do it here.

Let us explain the formulation of Theorem 1 by a particular example. According to Theorem 1, we have the following equality:

$$\frac{2x^3+x^2+x+2}{(x-1)^2(x^2+x+1)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{Mx+N}{x^2+x+1}, \tag{4}$$

where A_1, A_2, M, N are quite definite constant numbers. In order to find them, we reduce (4) to a common denominator and equate the numerators on its right-hand and left-hand sides:

$$2x^3 + x^2 + x + 2 = A_1 (x - 1)(x^2 + x + 1) + A_2 (x^2 + x + 1) + (Mx + N)(x - 1)^2. \tag{5}$$

Removing the parentheses in the right-hand side of (5), we collect the terms with the same powers of x and equate the coefficients of identical powers of x standing in both parts (see Sec. 4.14, Theorem 2):

$$\left.\begin{aligned} 2 &= A_1 + M, \\ 1 &= A_2 + N - 2M, \\ 1 &= A_2 + M - 2N, \\ 2 &= -A_1 + A_2 + N. \end{aligned}\right\} \tag{6}$$

We have obtained four linear equations in four unknowns: A_1, A_2, M, N. According to Theorem 1, this system has a unique solution. Solving system (6), we obtain $A_1 = 1$, $A_2 = +2$, $N = M = 1$, and therefore

$$\frac{2x^3+x^2+x+2}{(x-1)^2(x^2+x+1)} = \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{x+1}{x^2+x+1}. \tag{7}$$

In the general case, with the coefficients A, B, C found in (3), everything is ready for integration of the fraction P_m/Q_n: the indefinite integral of the left-hand member of (3) is equal to the sum of indefinite integrals of all the terms of the right-hand side plus a certain constant C. As it was noted above, we know how to compute integrals of any term of (3).

In the case of Example (7) we have:

$$\int \frac{2x^3+x^2+x+2}{(x-1)^2(x^2+x+1)}\,dx$$
$$= \int \frac{dx}{x-1} + 2\int \frac{dx}{(x-1)^2} + \int \frac{x+1}{x^2+x+1}\,dx = \ln|x-1|$$
$$-\frac{2}{x-1} + \ln\sqrt{x^2+x+1} + \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C.$$

Remark 1. Equality (5) is true for any $x \neq 1$. But then it is true for $x = 1$ as well, since both on the left

and on the right in (5) there stand continuous functions of x. Substituting $x = 1$ into (5), we get $6 = 3A_2$, i.e. $A_2 = 2$; and, setting $x = 0$, we obtain $2 = -A_1 + A_2 + N$, i.e. $N = A_1$. These results ($A_2 = 2, N = A_1$) considerably simplify system (6). In practical computations such reasons should not be ignored.

Remark 2. In principle, any rational function is integrable in elementary functions. Practically, complete integration of (1) can be effected if all the roots of Q_n and their multiplicities are known. But, as it was mentioned in Sec. 5.5, we cannot always succeed in finding all the roots. Due to this fact, any simplifications of the integral of rational fraction (1) are of great value.

In this connection, of great importance for practice is the simplification technique suggested by Ostrogradsky* which is usually discussed in more comprehensive textbooks**.

Sec. 5.7. Integration of Irrational Functions

It is very difficult to integrate an irrational elementary function even if the elementary function which is the integral of the given function is actually existent.

In this section we shall consider some cases in which, with the aid of substitution of a variable, integration of irrational functions is reduced to integration of rational functions (as we say, the integral is rationalized).

Let $R(x, y)$ be a rational function of two arguments, i.e. in order to obtain $R(x, y)$, only arithmetical operations are performed with x and y. For example,

$R(x, y) = \frac{axy + y^2}{cx + x^{10}y}$ is a rational function, and

$f(x, y) = \sqrt{x+y} + x^2$ is not a rational function.

I. Compute $\int R\left(x, \sqrt[m]{\frac{ax+b}{cx+d}}\right) dx$, where a, b, c, d

* Ostrogradsky, Mikhail Vasilievich (1801-1861). Prominent Russian mathematician.

** See, for instance, *A Course of Mathematical Analysis* by S.M. Nikolsky, vol. 1, § 8.7, Mir Publishers, Moscow, 1981.

are constant numbers, m is a natural number, $ad-bc\neq 0$, $R(x, y)$ is a rational function.

A function of the form $R\left(x, \sqrt[m]{\frac{ax+b}{cx+d}}\right)$ is called a *fractional linear irrational.*

Let us show that the substitution $t=\sqrt[m]{\frac{ax+b}{cx+d}}$ rationalizes the integral. Indeed, $t^m=\frac{ax+b}{cx+d}$, whence $x=\frac{b-dt^m}{ct^m-a}$ which is a rational function of t. Further,

$$dx=\frac{mt^{m-1}[ad-bc]}{(ct^m-a)^2}\,dt.$$

Therefore

$$\int R\left(x, \sqrt[m]{\frac{ax+b}{cx+d}}\right)dx$$

$$=\int R\left(\frac{b-dt^m}{ct^m-a}, t\right)\frac{mt^{m-1}[ad-bc]}{(ct^m-a)^2}\,dt=\int R_1(t)\,dt,$$

where $R_1(t)$ is a rational function of t which we know how to integrate.

Example 1. Compute $\int\frac{1}{(x-1)^2}\sqrt[3]{\frac{x+1}{x-1}}\,dx$. Here $R(x, y)=\frac{y}{(x-1)^2}$. Setting $\sqrt[3]{\frac{x+1}{x-1}}=t$, we obtain $x=\frac{t^3+1}{t^3-1}$, $dx=\frac{-6t^2\,dt}{(t^3-1)^2}$, $x-1=\frac{2}{t^3-1}$. Hence,

$$\int\frac{1}{(x-1)^2}\sqrt[3]{\frac{x+1}{x-1}}\,dx=\int t\,\frac{-6t^2}{(t^3-1)^2}\cdot\frac{(t^3-1)^2}{4}\,dt$$

$$=-\frac{3}{2}\int t^3\,dt=-\frac{3}{8}t^4+C=-\frac{3}{8}\left(\sqrt[3]{\frac{x+1}{x-1}}\right)^4+C.$$

Example 2.

$$\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}}=\int\frac{dx}{(\sqrt[6]{x})^2+(\sqrt[6]{x})^3}=(\sqrt[6]{x}=t)$$

$$=\int\frac{6t^5\,dt}{t^2+t^3}=6\int(t^2-t+1)\,dt-\ln|1+t|$$

$$=2t^3-3t^2+6t-\ln|1+t|+C,$$

II. Compute $\int R(x, \sqrt{ax^2+bx+c})\,dx$, where a, b, c are constant numbers. The function $R(x, \sqrt{ax^2+bx+c})$ will be called a *quadratic irrational.*

If the trinomial ax^2+bx+c has real roots x_1, x_2, then $ax^2+bx+c = a(x-x_1)(x-x_2)$ and

$$R\left(x, \sqrt{ax^2+bx+c}\right)$$
$$= R\left(x, (x-x_1)\sqrt{\frac{x-x_2}{x-x_1}a}\right) = R_1\left(x, \sqrt{\frac{x-x_2}{x-x_1}}\right),$$

and the problem is reduced to Case I.

Therefore we shall hold that ax^2+bx+c has no real roots and $a>0$. Then the integral can be rationalized with the aid of Euler's substitution*:

$$t = \sqrt{ax^2+bx+c} + x\sqrt{a}.$$

Hence, $ax^2+bx+c = t^2 - 2x\sqrt{a}t + ax^2$, i.e. $x = \dfrac{t^2-c}{2t\sqrt{a}+b}$ is a rational function of t. But then

$$\sqrt{ax^2+bx+c} = t - x\sqrt{a} = t - \frac{t^2-c}{2t\sqrt{a}+b}\sqrt{a}$$

is also a rational function of t. Therefore

$$\int R(x, \sqrt{ax^2+bx+c})\,dx = \int R_1(t)\,dt.$$

Remark. If $a<0$, and $c>0$ $(ax^2+bx+c \geqslant 0)$, then we can apply the substitution

$$\sqrt{ax^2+bx+c} = xt + \sqrt{c}.$$

Example 3. Compute $\int \sqrt{a^2+x^2}\,dx$.

The binomial x^2+a^2 has no real roots. Therefore we set

$$t = \sqrt{x^2+a^2} + x, \quad x^2+a^2 = t^2 - 2tx + x^2, \quad x = \frac{t^2-a^2}{2t}$$

and

$$\sqrt{x^2+a^2} = t - x = \frac{t^2+a^2}{2t}.$$

* This substitution may also be applied in the case of real roots for $a>0$ on the interval where $ax^2+bx+c \geqslant 0$.

Hence

$$x\sqrt{x^2+a^2}=\frac{t^4-a^4}{4t^2},\quad dx=\frac{t^2+a^2}{2t^2}\,dt.$$

By virtue of this,

$$\int\sqrt{x^2+a^2}\,dx=\int\frac{t^2+a^2}{2t}\cdot\frac{t^2+a^2}{2t^2}\,dt=\frac{1}{4}\int\left[t+\frac{2a^2}{t}+\frac{a^4}{t^3}\right]dt$$
$$=\frac{a^2}{2}\ln|t|+\frac{t^2}{8}-\frac{a^4}{8t^2}+C=\frac{a^2}{2}\ln|t|+\frac{t^4-a^4}{8t^2}+C$$
$$=\frac{a^2}{2}\ln|x+\sqrt{x^2+a^2}|+\frac{x}{2}\sqrt{x^2+a^2}+C.$$

III. Integration of expressions $R(\cos x, \sin x)$. The rationalization of $\int R(\cos x, \sin x)\,dx$ is achieved by means of the substitution $t=\tan(x/2)$ $(-\pi<x<\pi)$ which is called *universal*. Indeed,

$$\sin x=\frac{2\tan(x/2)}{1+\tan^2(x/2)}=\frac{2t}{1+t^2},\quad \cos x=\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}=\frac{1-t^2}{1+t^2},$$
$$x=2\arctan t,\quad dx=\frac{2dt}{1+t^2},$$

therefore

$$\int R(\cos x, \sin x)\,dx=\int R\left(\frac{1-t^2}{1+t^2},\frac{2t}{1+t^2}\right)\frac{2dt}{1+t^2}=\int R_1(t)\,dt.$$

If the function $R(x, y)$ possesses the properties of evenness or oddness with respect to x or y, then we can also use other substitutions for rationalizing the integral.

Let

$$R(u, v)=\frac{P(u, v)}{Q(u, v)}\quad (u=\cos x,\ v=\sin x),$$

where P and Q are polynomials of u and v.

(1) If one of the polynomials P, Q is even and the other is odd with respect to v, then the substitution $t=\cos x$ rationalizes the integral.

(2) If one of the polynomials P, Q is even and the other is odd with respect to u, then the substitution $t=\sin x$ rationalizes the integral.

(3) If P and Q: (a) both remain unchanged when replacing u and v by $-u$ and $-v$, respectively, or (b) both change sign, then the integral is rationalized by the substitution $t = \tan x$ (or $t = \cot x$).

Examples.

(1) $$\int \frac{dx}{\sin x} = \left(t = \tan \frac{x}{2}\right) = \int \frac{dt}{t} = \ln|t| + C = \ln\left|\tan \frac{x}{2}\right| + C.$$

(2) $$\int \frac{\sin^3 x}{\cos^4 x}\, dx = -\int \frac{\sin^2 x\, d(\cos x)}{\cos^4 x} = -\int \frac{1-\cos^2 x}{\cos^4 x}\, d(\cos x) = (t = \cos x) = -\int \frac{1-t^2}{t^4}\, dt.$$

In this case $R(u, v) = \frac{v^3}{u^4} = \frac{v^3}{u^4 v^0}$, i.e. the numerator is odd with respect to v and the denominator is even, which brings us to Case (1).

(3) $$\int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \int \frac{dx}{(a^2 + b^2\tan^2 x)\cos^2 x} = (t = \tan x) = \int \frac{dt}{a^2 + b^2t^2}.$$

Here the numerator $P(u, v) = 1$, and the denominator $Q(u, v) = a^2u^2 + b^2v^2$. Both of them remain unchanged when u and v are replaced by $-u$ and $-v$, respectively, i.e. we have Case 3 (a).

CHAPTER 6

THE DEFINITE INTEGRAL

Sec. 6.1. Some Problems of Geometry and Physics Leading to the Notion of the Definite Integral and Its Definition

(a) Let there be given a nonnegative continuous function $f(x)$ on a closed interval $[a, b]$ (a and b finite numbers). Its graph is represented in Fig. 75. Let us set the following problem: it is required to determine the notion of the area of a figure bounded by the curve $y = f(x)$, the x-axis and two straight lines $x = a$ and $x = b$, and to compute this area. It is natural to solve this problem in the following way.

Let us divide the interval $[a, b]$ into n parts (subintervals) by points of division

$$a = x_0 < x_1 < \ldots < x_n = b, \tag{1}$$

On choosing an arbitrary point ξ_j in each subinterval

$$[x_j, \ x_{j+1}] \quad (j = 0, 1, \ldots, n-1) \tag{2}$$

($\xi_j \in [x_j, x_{j+1}]$) and evaluating the function f at these points, we form the sum

$$S_n = \sum_{j=0}^{n-1} f(\xi_j)\,\Delta x_j \quad (\Delta x_j = x_{j+1} - x_j), \tag{3}$$

which is called the *integral sum* and which is, obviously, equal to the sum of areas of the hatched rectangles (see Fig. 75).

Let us now make all Δx_j tend to zero so that the greatest partition subinterval tends to zero. If in doing so S_n tends to a definite limit S which is independent of the ways of partition (1) and the choice of points ξ_j in the subintervals, then it is natural to call S the *area of our curvilinear figure.* Hence,

$$S = \lim_{\max \Delta x_j \to 0} \sum_{j=0}^{n-1} f(\xi_j)\,\Delta x_j. \tag{4}$$

Thus, we have defined the area of our curvilinear figure (trapezoid). There arises a question whether each such figure has an area, or, in other words, whether its integral sum S_n tends, in fact, to a finite limit as $\Delta x_j \to 0$. It will be proved later on that this question is answered in the affirmative: every above defined curvilinear figure

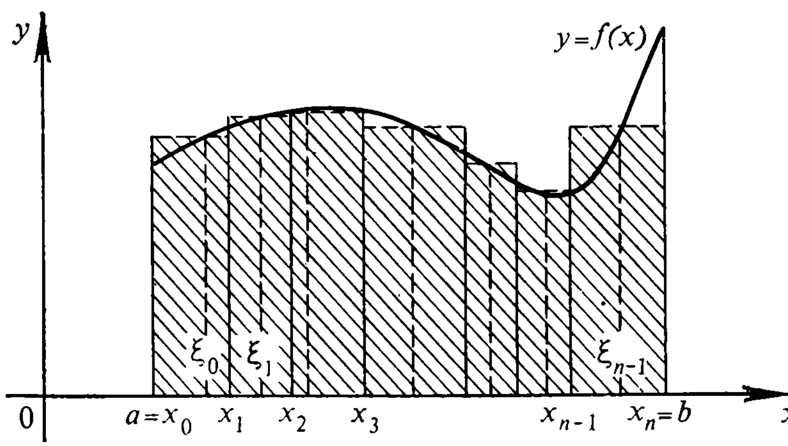

Fig. 75

corresponding to a certain continuous function $f(x)$ really has an area in the sense of the definition made which is thus representable by the number S dependent on this figure.

To what extent is the given definition of area natural? This question, as always in such cases, is answered by practice. We only want to say that practical solutions have justified this definition. We shall have many chances to get convinced that it is correct.

(b) Given a linear nonhomogeneous rod lying on the x-axis within the limits of a closed interval $[a, b]$. It is required to determine the mass of this rod. Let the density of mass distribution along the rod is a certain continuous function of x: $\rho(x)$.

In order to determine the mass of the given rod, let us divide it into n arbitrary parts by points $a = x_0 < < x_1 < \ldots < x_n = b$. Next we take in each subinterval $[x_i, x_{i+1}]$ an arbitrary point ξ_i.

Since the function $\rho(x)$ varies but little within the limits of $[x_i, x_{i+1}]$, we may regard the mass of the portion

of the rod corresponding to the subinterval $[x_i, x_{i+1}]$ as approximately equal to $\rho(\xi_i)\Delta x_i$, where $\Delta x_i = x_{i+1} - x_i$.

And the mass m of the entire rod is approximately equal to

$$\rho(\xi_0)\Delta x_0 + \rho(\xi_1)\Delta x_1 + \ldots + \rho(\xi_{n-1})\Delta x_{n-1} = \sum_{i=0}^{n-1} \rho(\xi_i)\Delta x_i.$$

The exact value of the mass is, obviously, obtained in the limit, when the greatest subinterval (that is, the subinterval with the maximum length) tends to zero, i.e.

$$m = \lim_{\max \Delta x_i \to 0} \sum_{i=0}^{n-1} \rho(\xi_i)\Delta x_i. \qquad (4')$$

The above considered problems (one of them belonging to geometry, the other to physics) lead us to one and the same mathematical operation on functions of different origin defined on the interval $[a, b]$. We shall encounter many other concrete problems whose solution is reduced to a similar operation on a function given on a closed interval. This operation is called the *operation of integration of a function on a closed interval*, and its result (i.e. a number) is called the *definite integral of a function on a closed interval.*

Definition 1. *Let there be given a function f on a closed interval $[a, b]$. Let us divide the interval $[a, b]$ into n parts with the aid of arbitrary points*

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b.$$

Such a construction will be referred to as a partition R of the interval $[a, b]$. We then choose in each subinterval $[x_j, x_{j+1}]$ an arbitrary point $\xi_j \in [x_j, x_{j+1}]$ and form the sum

$$\sigma_R = \sigma_R(f) = \sum_{j=0}^{n-1} f(\xi_j)\Delta x_j \quad (\Delta x_j = x_{j+1} - x_j),$$

called an integral sum for the function f corresponding to the partition R.

The maximum length of the subintervals $[x_j, x_{j+1}]$ *of the partition* R *will be denoted as*

$$\lambda_R = \max_{0 \leqslant j \leqslant n-1} \Delta x_j$$

The limit (provided it exists) to which the integral sum σ_R *tends, as* $\lambda_R \to 0$, *is called the definite integral of the function* f *on a closed interval* $[a, b]$ *and is denoted in the following way*

$$\lim_{\lambda_R \to 0} \sigma_R = \lim_{\max \Delta x_j \to 0} \sum_{j=0}^{n-1} f(\xi_j) \Delta x_j = \int_a^b f(x)\, dx \quad (a < b). \quad (5)$$

The numbers a and b are called, respectively, the *lower* and the *upper limit of integration**.

Definition 1 is equivalent to the following.

Definition 1′. *The definite integral of a function* f *on a closed interval* $[a, b]$ *is defined as a number* I *satisfying the following property: given any* $\varepsilon > 0$, *there is a number* $\delta > 0$ *such that*

$$|\sigma_R - I| = \left| \sum_{j=0}^{n-1} f(\xi_j) \Delta x_j - I \right| < \varepsilon$$

for any partition R *of the interval* $[a, b]$ *in which*

$$\lambda_R = \max_j \Delta x_j < \delta$$

for any choice of points $\xi_j \in [x_j, x_{j+1}]$.

The definite integral is the fundamental concept of the integral calculus. It was introduced (in the above way) by Cauchy for continuous functions and (in the general case) by Riemann** for functions not necessarily continuous, i.e. for Riemann integrable functions. Limit (5) is usually called the *Riemann integral*, and the function for which this limit exists is said to be *Riemann integrable* (in the proper sense).

If a function f is continuous on $[a, b]$, then, as we shall learn later on, for it there always exists limit (5).

---

* It is unfortunate that the word "limit" is used in this connection. The limits of integration have nothing to do with the limits considered in Chapter 3 (Sec. 3.2)—*Tr.*

** Riemann, Georg Friedrich Bernhard (1826-1866). Distinguished German mathematician.

A function continuous on a closed interval $[a, b]$ is said *Cauchy integrable* on this interval.

In (a) we defined (see Fig. 75) the area of a plane figure bounded by the graph of a continuous function $y = f(x) \geqslant 0$ (from above), by the x-axis (from below) and by two straight lines $x = a$ and $x = b$ (from the sides). Now we can say that the area of this figure is equal to the definite integral of f on $[a, b]$:

$$S = \int_a^b f(x)\,dx.$$

We may also say that the mass of the rod dealt with in (b) is equal to the definite integral of its line density $\rho(x)$ within the limits of $[a, b]$:

$$m = \int_a^b \rho(x)\,dx.$$

Thus, briefly stated, the definition of the definite integral reads as follows: *the definite integral of a function f on $[a, b]$ is the limit of integral sum* (5) *when the length of the greatest subinterval of partition R tends to zero.*

In this definition, which is no longer related with the problem of finding the area, the function f is not necessarily continuous and nonnegative on $[a, b]$. It should be noted that this definition does not assert the existence of the definite integral for any function f defined on $[a, b]$, that is, the existence of limit (5). It only states that if this limit exists for a function f defined on $[a, b]$, then it is called the definite integral of f on $[a, b]$.

It should also be borne in mind that when we say that the limit I exists, we mean that it is independent both of the ways in which the interval is partitioned into subintervals and of the choice of the points ξ_j on the subintervals thus obtained.

Direct computation of the definite integral by formula (5) turns out to be difficult, since the integral sums of any composite functions have a cumbersome form and it is often rather difficult to transform them to a form convenient for computing the limits. Unfortunately,

mathematicians have not yet succeeded in developing some general methods for this purpose. It is of interest to note that Archimedes was the first to solve a problem of such kind. Reasoning in the way remotely resembling the modern method of limits, he calculated the area of a parabola segment. Later on, during centuries many mathematicians were busy solving problems on computing the areas of various figures and volumes of solids. But up to the seventeenth century both the statement of such problems and questions and the methods of their solution were of especially particular character. Newton* and Leibniz** contributed greatly in this field by developing a general method for solving such problems. They showed that the computation of the definite integral of a function can be reduced to finding its antiderivative.

As was noted above, a function continuous on a closed interval $[a, b]$ is integrable on this interval. This will be proved in Sec. 6.7.

It will also be proved that a function monotone on a closed interval $[a, b]$ is integrable on this interval. It should be taken into consideration that a monotone function can have discontinuities in a finite or even countable number (see Theorem 2 proved in Sec. 3.4).

Figure 76 represents the graph of a function $y = f(x)$ defined on a closed interval $[0, a]$. This function is continuous on $[0, x_1]$, decreases on $[x_1, x_2]$ and increases on $[x_2, a]$. Consequently, it is integrable on each of these subintervals. But then, on the basis of additive properties of the integral to be discussed below, our function is integrable throughout the entire interval $[0, a]$ (see Sec. 6.2, Theorem 3).

Thus, if the interval $[a, b]$ on which a function $y = f(x)$ is defined can be cut into a finite number of subintervals on each of which it is continuous or monotone, then this function is integrable on $[a, b]$.

---

* Newton, Sir Isaac (1642-1727). Superb English mathematician, physicist, and astronomer. With Aristotle and Gauss, one of the three greatest mathematicians of all time. He and Leibniz invented calculus independently.

** Leibniz, Gottfried Wilhelm von (1646-1716). Great German mathematician and philosopher.

Newton and Leibniz proved the theorem relating two important notions of mathematical analysis—the integral and the derivative. This theorem is expressed by the following relationship (the Newton-Leibniz formula):

$$F(b) - F(a) = \int_a^b f(x)\,dx. \tag{6}$$

Here $f(x)$ is an arbitrary function continuous on $[a, b]$ and $F(x)$ is some of its antiderivatives on $[a, b]$ ($F'(x) = f(x)$).

Thus, *in order to evaluate the definite integral of a continuous function f on a closed interval $[a, b]$, it is necessary*

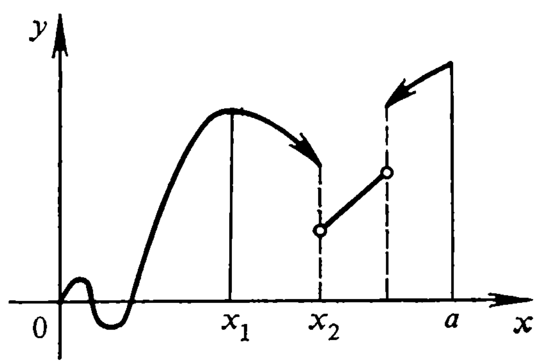

Fig. 76

to find its antiderivative $F(x)$ and take the difference $F(b) - F(a)$ of the values of this antiderivative at the end points of the interval $[a, b]$.

If it is regarded as known that a continuous on $[a, b]$ function $f(x)$ is integrable on this interval and that there exists an antiderivative $F(x)$ of this function, then formula (6) can be readily derived.

Let R be an arbitrary partition

$$a = x_0 < x_1 < \ldots < x_n = b$$

of the interval $[a, b]$ into subintervals. Then (the explanations are given below)

$$F(b)-F(a)=F(x_n)-F(x_0)=F(x_n)-F(x_{n-1})+F(x_{n-1})$$

$$-\ldots-F(x_1)+F(x_1)-F(x_0)=\sum_{k=0}^{n-1}[F(x_{k+1})-F(x_k)]$$

$$=\sum_{k=0}^{n-1}F'(\xi_k)(x_{k+1}-x_k)=\sum_{k=0}^{n-1}f(\xi_k)\Delta x_k$$

$$\xrightarrow[\lambda_R\to 0]{}\int_a^b f(x)\,dx, \qquad (7)$$

whence formula (6) follows.

To the fourth equality of (7) we have applied Lagrange's mean-value theorem

$$F(x_{k+1})-F(x_k)=F'(\xi_k)(x_{k+1}-x_k),$$

by which ξ_k is a certain point of the interval (x_k, x_{k+1}). The last relationship follows from the fact that the function f is continuous on $[a, b]$ and, consequently, integrable on $[a, b]$, and therefore any of its integral sums (including the one obtained by applying Lagrange's theorem) tends to the definite integral of f on $[a, b]$ as $\lambda_R \to 0$.

The following theorem holds true.

Theorem 1. *A function unbounded on a closed interval* $[a, b]$ *is not integrable on this interval.*

Thus, *for a function f to be integrable on a closed interval* $[a, b]$, *it is necessary that it be bounded on this interval.*

But this condition is not sufficient.

Example. The function

$$\psi(x)=\begin{cases}1, & \text{if } x \text{ is rational,}\\ -1, & \text{if } x \text{ is irrational,}\end{cases}$$

is bounded: $|\psi(x)|=1$, but not integrable on any interval $[a, b]$ $(a<b)$.

Indeed, if rational numbers are chosen in its integral sum as points ξ_j, then

$$\sigma_R=\sum_{j=0}^{n-1}\psi(\xi_j)\,\Delta x_j=\sum_{j=0}^{n-1}1\cdot\Delta x_j=b-a.$$

If ξ_j are chosen to be irrational, then

$$\sigma_R = \sum_{j=0}^{n-1} (-1)\,\Delta x_j = -(b-a).$$

This shows that σ_R cannot have one and the same limit for any choice of ξ_j, and, consequently, the function ψ is not integrable on $[a, b]$.

The Proof of Theorem 1. Let

$$\sigma_R = \sum_{i=0}^{n-1} f(\xi_i)(x_{i+1} - x_i)$$

be an integral sum of the function f corresponding to a certain partition R: $a = x_0 < x_1 < \ldots < x_n = b$. If we assume that the function f is not bounded on $[a, b]$, then it is necessarily unbounded on one of the subintervals, say, on $[x_{i0}, x_{i0+1}]$. Let us fix $\xi_i \in [x_i, x_{i+1}]$ for all $i \neq i_0$, and ξ_{i0} will be regarded for the time being as variable. The term $f(\xi_{i_0})(x_{i_0+1} - x_{i_0})$ is not bounded on $[x_{i_0}, x_{i_0+1}]$, and the sum of the remaining terms is a definite number. But then $|\sigma_R|$ can be made arbitrarily large with the point $\xi_{i_0} \in [x_{i_0}, x_{i_0+1}]$ appropriately chosen, and the function f cannot be integrable on $[a, b]$. But from the integrability of the function f on $[a, b]$ it follows that its integral sums are bounded for any choice of ξ_i.

However, later on we shall introduce the notion of an improper integral. Some functions, unbounded on a closed interval, are integrable in the improper sense. But this will be discussed below.

Sec. 6.2. Properties of Definite Integrals

In this section we shall study the properties of integrable functions. As was noted above, functions, continuous and monotone on a closed interval $[a, b]$, are integrable on this interval. This will be proved in Sec. 6.7.

Theorem 1. *If M is a constant, then*

$$\int_a^b M\,dx = M(b-a). \tag{1}$$

Indeed, for any partition R of a closed interval $[a, b]$ the integral sum of the function $f(x) = M$ is equal to

$$\sigma_R = \sum_{j=0}^{n-1} M\,\Delta x_j = M \sum_{i=0}^{n-1} \Delta x_j = M(b-a).$$

Hence,

$$\lim_{\lambda_R \to 0} \sigma_R = M\,(b-a).$$

Theorem 2. *For the function*

$$\psi_c(x) = \begin{cases} 0, & x \in [a, b], \quad x \neq c, \\ A, & x = c, \end{cases}$$

$$\int_a^b \psi_c(x)\,dx = 0.$$

Indeed, let there be given an arbitrary partition R of the interval $[a, b]$:

$$a = x_0 < x_1 < \ldots < x_n = b.$$

One of the half-intervals of this partition, say $[x_m, x_{m+1})$ contains the point c: $x_m \leqslant c < x_{m+1}$. Therefore the integral sum

$$\sigma_R(\psi_c) = \sum_{k=0}^{n-1} \psi_c(\xi_k)\,\Delta x_k = \psi_c(\xi_{m-1})\,\Delta x_{m-1} + \psi_c(\xi_m)\,\Delta x_m$$

(the remaining terms being automatically equal to zero). Since $|\psi_c(x)| \leqslant |A|$, we have

$$|\sigma_R(\psi_c)| \leqslant |A| \quad (\Delta x_{m-1} + \Delta x_m) \to 0$$

as $\lambda_R \to 0$, and the theorem has been proved.

Theorem 3. *If a function f is integrable on each of the intervals $[a, c]$, $[c, b]$ $(a < c < b)$, then it is integrable on $[a, b]$ and*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{2}$$

(*additive property of the definite integral*).

Proof. Let there be given an arbitrary partition of the interval $[a, b]$

$$R:\ a = x_0 < x_1 < \ldots < x_n = b,$$

but such that one of the points of R, let it be x_m, coincides with the point c $(x_m = c)$. Then R induces on the intervals $[a, c]$ and $[c, b]$ certain partitions R_1 and R_2:

$$R_1:\ a = x_0 < x_1 < \ldots < x_m = c,$$

$$R_2:\ c = x_m < x_{m+1} < \ldots < x_n = b$$

and

$$\sigma_R = \sum_{j=0}^{n-1} f(\xi_j)\,\Delta x_j = \sum_{j=0}^{m-1} f(\xi_j)\,\Delta x_j + \sum_{j=m}^{n-1} f(\xi_j)\,\Delta x_j = \sigma_{R_1} + \sigma_{R_2},$$

i.e. $\sigma_R = \sigma_{R_1} + \sigma_{R_2}$. Let

$$\lambda_R = \max_{0 \leqslant j \leqslant n-1} |\Delta x_j| \to 0.$$

Then, a fortiori, $\lambda_{R_1} \to 0$ and $\lambda_{R_2} \to 0$. Hence,

$$\lim_{\lambda_R \to 0} \sigma_R = \lim_{\lambda_{R_1} \to 0} \sigma_{R_1} + \lim_{\lambda_{R_2} \to 0} \sigma_{R_2} = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

This equality is proved for the time being for partitions R containing a point c. But then it is true for any partitions R as well (see Lemma 1 below). Consequently, the integral $\int_a^b f(x)\,dx$ exists and (2) holds.

By definition,

$$\int_a^a f(x)\,dx = 0, \tag{3}$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx, \quad b < a, \tag{4}$$

where f is integrable on $[b, a]$.

Taking into account (3) and (4), it is easy to see that equality (2) is true for any numbers a, b, c, provided f is integrable on the greatest of the intervals $[a, b]$, $[a, c]$, $[c, b]$.

For instance, if $c < a < b$, then, by Theorem 3,

$$\int_c^b f\,dx = \int_c^a f\,dx + \int_a^b f(x)\,dx$$

or

$$\int_a^b f\,dx = \int_c^b f\,dx - \int_c^a f\,dx = \int_a^c f\,dx + \int_c^b f\,dx,$$

and we have obtained (2).

Theorem 4. *If functions f_1 and f_2 are integrable on $[a, b]$ and A, B are arbitrary numbers, then*

$$\int_a^b (Af_1 + Bf_2)\,dx = A\int_a^b f_1\,dx + B\int_a^b f_2\,dx. \tag{5}$$

In particular, for $B = 0$ we get the equality

$$\int_a^b Af_1\,dx = A\int_a^b f_1\,dx, \tag{6}$$

showing that a constant factor in the integrand can be taken outside the sign of the definite integral.

For $A = 1$, $B = \pm 1$ we obtain

$$\int_a^b (f_1 \pm f_2)\,dx = \int_a^b f_1\,dx \pm \int_a^b f_2\,dx. \tag{7}$$

Proof. For an arbitrary partition R we have

$$\sum_{j=0}^{n-1} [Af_1(\xi_j) + Bf_2(\xi_j)]\,\Delta x_j$$

$$= A\sum_{j=0}^{n-1} f_1(\xi_j)\,\Delta x_j + B\sum_{j=0}^{n-1} f(\xi_j)\,\Delta x_j.$$

Hence, passing to the limit as $\lambda_R \to 0$, we obtain equality (5) which is, obviously, also true for $b \leqslant a$.

Theorem 5. *If an integrable on $[a, b]$ function f is modified at a point $c \in [a, b]$, then for the function f_1 obtained as a result of the modification the following equality takes place:*

$$\int_a^b f_1(x)\,dx = \int_a^b f(x)\,dx.$$

Proof. The modification of the function f only at the point c is reduced to the following: we add to $f(x)$ a function of the form

$$\psi_c(x) = \begin{cases} 0, & x \in [a, b], \quad x \neq c, \\ A, & x = c, \end{cases}$$

where A is a certain number. Then

$$f_1(x) = f(x) + \psi_c(x),$$

and, by Theorem 2,

$$\int_a^b \psi_c(x)\, dx = 0.$$

Therefore, by virtue of Theorem 4,

$$\int_a^b f_1(x)\, dx = \int_a^b f(x)\, dx + \int_a^b \psi_c(x)\, dx = \int_a^b f(x)\, dx,$$

which was to be proved.

Remark 1. We see from Theorem 5 that the integrability of the function f is independent of the values attained by it at a certain definite point.

For instance, the function $\psi(x) = (\sin x)/x$ is defined on the half-interval $(0, 1]$. If we make it to be equal to 1 for $x = 0$ $(\psi(0) = 1)$, then it will be continuous and, consequently, integrable on the closed interval $[0, 1]$. But it will remain integrable and its integral $\int_0^1 \psi(x)\, dx$ will be equal to the same value if we assume that $\psi(0) = A$, where A is any number.

Theorem 6. *If two functions f and φ are integrable on a closed interval $[a, b]$ and satisfy the inequality*

$$f(x) \leqslant \varphi(x),$$

on this interval, then

$$\int_a^b f(x)\, dx \leqslant \int_a^b \varphi(x)\, dx \quad (a \leqslant b). \tag{8}$$

Proof. For any partition R

$$\sum_{j=0}^{n-1} f(\xi_j)\,\Delta x_j \leqslant \sum_{j=0}^{n-1} \varphi(\xi_j)\,\Delta x_j$$

since $\Delta x_j > 0$. Therefore, on passing to the limit as $\lambda_R \to 0$, we obtain (8).

Theorem 7. *The following inequality is valid:*

$$\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx \quad (a \leqslant b) \tag{9}$$

or, if a is not necessarily less than b, then

$$\left|\int_a^b f(x)\,dx\right| \leqslant \left|\int_a^b |f(x)|\,dx\right|, \tag{9'}$$

if f and $|f|$ are integrable on $[a, b]$.

Proof. Obviously,

$$-|f(x)| \leqslant f(x) \leqslant |f(x)|, \quad \forall x \in [a, b].$$

But then, by Theorem 6

$$\int_a^b (-|f(x)|)\,dx \leqslant \int_a^b f\,dx \leqslant \int_a^b |f|\,dx, \quad (a < b)$$

or

$$-\int_a^b |f|\,dx \leqslant \int_a^b f\,dx \leqslant \int_a^b |f|\,dx,$$

or

$$\left|\int_a^b f\,dx\right| \leqslant \int_a^b |f|\,dx \quad (a < b),$$

which is what we set out to prove.

For $a < b$ the right-hand members of (9) and (9′) are equal to each other. And if $b < a$, then, by virtue of (4)

$$\left|\int_a^b f\,dx\right| = \left|\int_b^a f\,dx\right| \leqslant \int_b^a |f|\,dx = \left|\int_a^b |f|\,dx\right|,$$

i.e. (9′) takes place.

Finally, the case $a = b$ is reduced to the obvious relationship $0 \leqslant 0$. This proves (9).

Remark 2. The integrability of f on $[a, b]$ implies the integrability of $|f|$ on $[a, b]$ (see below Sec. 6.7, Remark 2). For concrete functions this is always obvious. For instance, if the function f is piecewise continuous on $[a, b]$ (as it will be proved, it is integrable), then $|f|$ is also piecewise continuous.

Conversely, the integrability of $|f(x)|$, generally speaking, does not imply the integrability of $f(x)$ on $[a, b]$.

For example, the function $\psi(x)$ (see the example given in the preceding section)

$$\psi(x) = \begin{cases} 1 & \text{for rational } x\text{'s}, \\ -1 & \text{for irrational } x\text{'s}, \end{cases}$$

is not integrable on $[a, b]$. Meanwhile, $|\psi(x)| = 1$ on $[a, b]$ and is a function integrable on $[a, b]$.

Theorem 8. *If a function f is integrable and nonnegative on $[a, b]$ and there exists a point $c \in [a, b]$ of continuity of f for which $f(c) > 0$, then*

$$\int_a^b f(x)\,dx > 0 \quad (a < b). \tag{10}$$

Proof. We shall assume that $c \in (a, b)$. Since f is continuous at the point c and $f(c) > 0$, there exists a closed interval $[c - \delta, c + \delta]$ such that (see Sec. 3.3, Theorem 4)

$$f(x) > \frac{f(c)}{2} = \eta > 0, \quad \forall\, x \in [c-\delta,\ c+\delta].$$

Then

$$\int_a^b f(x)\,dx = \int_a^{c-\delta} f(x)\,dx + \int_{c-\delta}^{c+\delta} f(x)\,dx + \int_{c+\delta}^b f(x)\,dx > 0,$$

since

$$\int_a^{c-\delta} f(x)\,dx \geqslant 0, \qquad \int_{c+\delta}^b f(x)\,dx \geqslant 0,$$

$$\int_{c-\delta}^{c+\delta} f(x)\,dx \geqslant \int_{c-\delta}^{c+\delta} \eta\,dx = 2\delta\eta > 0.$$

If $c = a$ or $c = b$, then, instead of $[c - \delta, c + \delta]$, we have to consider the interval $[a, a + \delta]$ or $[b - \delta, b]$, respectively.

Lemma 1. *Let R_* denote an arbitrary partition of a closed interval $[a, b]$ containing a point c as a point of division.*

A function f is bounded on $[a, b]$ and for its integral sums, corresponding only to partitions of the form R_*, we have

$$\lim_{\lambda_{R_*} \to 0} \sigma_{R_*} = I.$$

Then the function f is integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = I = \lim_{\lambda_R \to 0} \sigma_R.$$

Proof. Let R be an arbitrary partition of the interval $[a, b]$ not containing the point c:

$$R\colon a = x_0 < x_1 < \ldots < x_m < x_{m+1} < \ldots < x_n = b,$$

where $x_m < c < x_{m+1}$.

Adding the point c to R, we get R_*. If $\lambda_R \to 0$, then also $\lambda_{R_*} \to 0$.

Rejecting the term $f(\xi_m)(x_{m+1} - x_m)$ from σ_R and adding $f(\xi_m')(c - x_m) + f(\xi_m'')(x_{m+1} - c)$, we obtain the integral sum σ_{R_*}. It is clear that

$$\sigma_R = \sigma_{R_*} + \mu,$$

where $\mu = f(\xi_m)(x_{m+1} - x_m) - f(\xi_m')(c - x_m) - f(\xi_m'')(x_{m+1} - c)$, $x_m \leqslant \xi_m' \leqslant c$, $c \leqslant \xi_m'' \leqslant x_{m+1}$. Obviously,

$$|\mu| \leqslant M(x_{m+1} - x_m) + M(c - x_m) + M(x_{m+1} - c) = 2M(x_{m+1} - x_m) \xrightarrow[\lambda_R \to 0]{} 0.$$

Consequently,

$$\lim_{\lambda_R \to 0} \sigma_R = \lim_{\lambda_{R_*} \to 0} \sigma_{R_*} + \lim_{x_{m+1} - x_m \to 0} \mu = I + 0 = I.$$

Sec. 6.3. Integral as Function of Its Upper Limit

Let f be an integrable function defined on a closed interval $[a, b]$. It should be noted that

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du.$$

This means that the value of the definite integral does not depend on the notation of the variable of integration,

i.e. its value does not depend on the letter (x or u) written under the sign of f in the definite integral of f over the interval $[a, b]$, since in both cases any integral sum of f has the form

$$\sigma_R = \sum_{j=0}^{n-1} f(\xi_j)\,\Delta x_j.$$

If the given function f is integrable on $[a, b]$, then the integral of f on $[a, x]$ also exists for any x satisfying the inequalities $a \leqslant x \leqslant b$.

This assertion requires a proof, but we shall not prove it. As a rule, this assertion is obvious in particular cases.

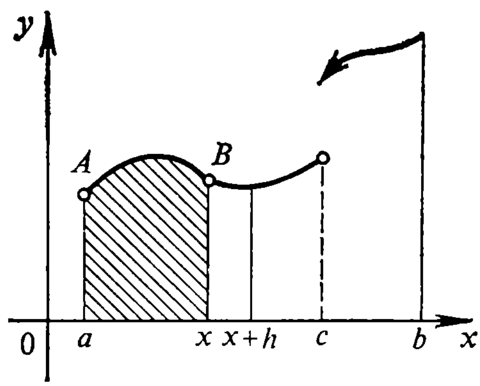

Fig. 77

For instance, a function, continuous (monotone) on a closed interval $[a, b]$ is also continuous (monotone) on $[a, x]$ and, consequently, integrable on $[a, x]$.

Let us take an arbitrary value $x \in [a, b]$. We shall be interested in the definite integral of f on $[a, x]$, which is a certain function of x. Let us denote it by $F(x)$.

Hence,

$$F(x) = \int_a^x f(u)\,du. \tag{1}$$

We use here the letter u as the variable of integration in order to distinguish it from the upper limit of integration x.

Depicted in Fig. 77 is the graph of a bounded piecewise continuous function f with a point of discontinuity c.

The number $F(x)$ for the given x is represented in the figure by the area of the figure $ABxa$. $F(x)$ changes as x varies on $[a, b]$.

Theorem 1. *If a function f is integrable on a closed interval $[a, b]$, then the function F defined by formula* (1) *is continuous at any point $x \in [a, b]$.*

Proof. Let us take an arbitrary point x and assign an increment h to it (shown in Fig. 77 is a positive h). We have

$$|F(x+h)-F(x)| = \left| \int_a^{x+h} f(u)\,du - \int_a^x f(u)\,du \right|$$

$$= \left| \int_x^{x+h} f(u)\,du \right| \leqslant M|h|$$

$$(M \geqslant |f(u)|, \quad \forall u \in [a, b]).$$

We have obtained the inequality

$$|F(x+h)-F(x)| \leqslant M|h|,$$

whence it follows:

$$\lim_{h\to 0} [F(x+h)-F(x)] = 0,$$

i.e. F is continuous at the point x.

It should be underlined that x may turn out to be either a point of continuity, or a point of discontinuity of f, but all the same, the function $F(x)$ is continuous at this point.

Theorem 2. *If an integrable on $[a, b]$ function f is continuous at a point $x \in [a, b]$, then at this point there exists a derivative* of F (see (1)):

$$F'(x) = f(x). \tag{2}$$

Proof. Let x be a point of continuity of f. We have

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left[\int_a^{x+h} f(u)\,du - \int_a^x f(u)\,du \right]$$

$$= \frac{1}{h} \int_x^{x+h} f(u)\,du = \frac{1}{h} \int_x^{x+h} \{f(x) + [f(u)-f(x)]\}\,du$$

$$= \frac{1}{h} f(x) h + \frac{1}{h} \int_{x}^{x+h} [f(u) - f(x)] \, du$$

$$= f(x) + \frac{1}{h} \int_{x}^{x+h} [f(u) - f(x)] \, du. \qquad (3)$$

In deriving (3) we utilized the above proved properties of the definite integral. In the fourth equality we took advantage of the fact that $f(x)$ is independent of u and, when integrating with respect to u, $f(x)$ should be regarded as a constant factor (see Theorem 1 proved in Sec. 6.2).

Let us prove that

$$\frac{1}{h} \int_{x}^{x+h} [f(u) - f(x)] \, du \xrightarrow[h \to 0]{} 0. \qquad (4)$$

The function f is continuous at the point x, therefore for any $\varepsilon > 0$ there is $\delta > 0$, such that if $|h| < \delta$, then

$$|f(u) - f(x)| < \varepsilon, \quad \forall u \in [x, x+h].$$

Therefore for $|h| < \delta$

$$\left| \frac{1}{h} \int_{x}^{x+h} [f(u) - f(x)] \, du \right| \leqslant \left| \frac{1}{h} \int_{x}^{x+h} |f(u) - f(x)| \, du \right|$$

$$\leqslant \left| \frac{1}{h} \int_{x}^{x+h} \varepsilon \, du \right| = \left| \frac{1}{h} \varepsilon h \right| = \varepsilon,$$

and we have justified property (4).

From (3), passing to the limit as $h \to 0$, on the basis of (4), we obtain that the derivative $F'(x)$ does exist:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

This proves Theorem 2.

Let us draw our attention to that in Theorem 2, although the function f was allowed to be discontinuous on $[a, b]$, but it was supposed, that at the point x at which the existence of the derivative of F was asserted,

the function f is continuous. Otherwise, the theorem would be, generally speaking, incorrect.

Theorem 2, in particular, states that if $f(x)$ is continuous on a closed interval $[a, b]$, then $F(x)$ has a derivative on this interval equal to $f(x)$ $(F'(x) = f(x), \forall x \in \in [a, b])$.

Thus, *if a function f is continuous on a closed interval $[a, b]$, then for it there exists an antiderivative on this interval. And integral* (1) *may be taken as one of the antiderivatives.*

Hence it follows that the *indefinite integral of a function f continuous on $[a, b]$, is equal to*

$$\int f(x)\,dx = \int_a^x f(u)\,du + C, \quad x \in [a, b],$$

where C is a certain constant.

Sec. 6.4. The Newton-Leibniz Formula

This formula has the form:

$$\int_a^b f(u)\,du = \Phi(b) - \Phi(a) = \Phi(x)\Big|_{x=a}^{x=b}. \qquad (1)$$

Here $f(u)$ is a function continuous on a closed interval $[a, b]$, and $\Phi(u)$ is its arbitrary antiderivative on that interval.

The Newton-Leibniz formula was proved in Sec. 6.1. It was assumed there as known that a function f, continuous on $[a, b]$, is integrable and has an antiderivative on $[a, b]$.

Now we know from the preceding section that the integrability of a function continuous on $[a, b]$ implies the existence of its antiderivative on $[a, b]$.

Let us give another proof of the Newton-Leibniz formula. Returning to the function

$$F(x) = \int_a^x f(u)\,du, \qquad (2)$$

we note that

$$F(a)=\int_a^a f(u)\,du=0 \quad \text{and} \quad F(b)=\int_a^b f(u)\,du. \qquad (3)$$

Besides, we know that $F(x)$ is an antiderivative of $f(x)$ on $[a, b]$. Therefore, if $\Phi(x)$ is some other antiderivative, then there exists a constant C such that

$$\Phi(x) = F(x) + C, \quad \forall x \in [a, b]. \qquad (4)$$

From (2), (3), (4), we obtain

$$\Phi(b)-\Phi(a)=F(b)-F(a)=\int_a^b f(u)\,du,$$

and we have proved formula (1).

Example 1.

$$\int_0^1 x^2\,dx=\frac{x^3}{3}\bigg|_{x=0}^{x=1}=\frac{1}{3}.$$

This shows that the area (Fig. 78) of the hatched figure lying below the parabola $y = x^2$ is equal to 1/3.

Example 2.

$$\int_0^{\pi} \sin x\,dx=-\cos x\bigg|_0^{\pi}=1+1=2.$$

Thus, the area of the figure (Fig. 79) bounded by the sinusoid $y = \sin x$ and the x-axis is equal to 2.

Example 3. The function

$$\varphi(x)=\operatorname{sgn} x=\begin{cases} -1, & -1\leqslant x<0, \\ 0, & x=0, \\ 1, & 0<x\leqslant 1, \end{cases}$$

is continuous on a closed interval $[-1, 1]$ except for the point $x = 0$. The interval $[-1, 1]$ can be cut into two subintervals: $[-1, 0]$ and $[0, 1]$ on which the given function is monotone and, consequently, integrable. Therefore, $\varphi(x)$ is integrable on $[-1, 1]$. The following

formula is valid:

$$F(x)=\int_{-1}^{x} \operatorname{sgn} u \, du = -1+|x| \quad (-1 \leqslant x \leqslant 1). \tag{5}$$

Indeed, on the half-interval $[-1, 0)$ the function $\varphi(x)$ is continuous: $\varphi(x) = -1$. Its antiderivative on this

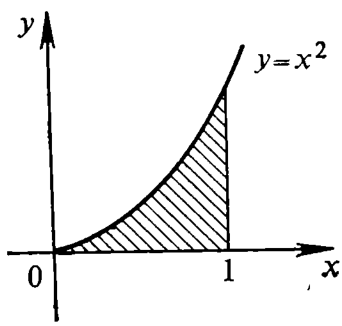

Fig. 78

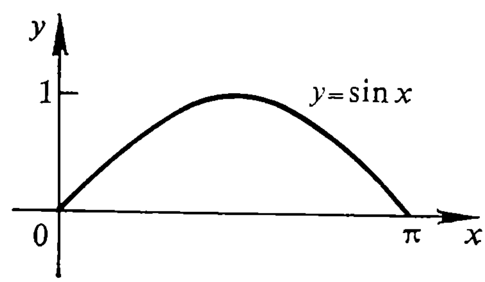

Fig. 79

half-interval is equal to $-x$. Therefore, applying the Newton-Leibniz formula, we get

$$\int_{-1}^{x} \operatorname{sgn} u \, du = \int_{-1}^{x} (-1)\, du = -u \Big|_{-1}^{x}$$

$$= -1-x \quad (-1 \leqslant x < 0). \tag{6}$$

By virtue of Theorem 1, $F(x)$ is continuous, in particular, at the point $x = 0$, therefore

$$F(0) = \lim_{x \to 0} (-1 - x) = -1. \tag{7}$$

For $x > 0$

$$F(x) = \int_{-1}^{x} \operatorname{sgn} u \, du = \int_{-1}^{0} \operatorname{sgn} u \, du + \int_{0}^{x} 1 \cdot du$$

$$= -1 + u \Big|_{0}^{x} = -1 + x. \tag{8}$$

From (6), (7), (8) there follows (5).

By integrating from the point $x = 0$, we obtain a more elegant formula:

$$\int_0^x \operatorname{sgn} u \, du = |x|. \tag{9}$$

Under the integral sign in (9) there stands a bounded function discontinuous at the point $x = 0$. The integral as a function of the upper limit $F(x) = |x|$ is a continuous function, at the point $x = 0$ as well, which agrees with Theorem 1 of the preceding section. But the derivative $F'(0)$ is not existent, and this does not contradict Theorem 2 from the same section which guarantees the existence of the derivative $F'(x)$ only if f is continuous at the point x.

Theorem 1 (on Integration by Change of Variable). *The following equality takes place*:

$$\int_a^b f(x)\, dx = \int_c^d f[\varphi(t)]\, \varphi'(t)\, dt, \tag{10}$$

where the function $\varphi(t)$ *is continuously differentiable on* $[c, d]$, $a = \varphi(c)$, $b = \varphi(d)$ *and* $f(x)$ *is continuous on* $[A, B] = \varphi([c, d])$, *that is on the image of the interval* $[c, d]$ *with the aid of the function* φ.

Proof. Let $F(x)$ and $\Phi(t)$ be antiderivatives of $f(x)$ and $f[\varphi(t)]\,\varphi'(t)$, respectively. Then (see Sec. 5.2, (1), *et al.*) the following identity holds true: $\Phi(t) = F[\varphi(t)] + C$, $c \leqslant t \leqslant d$, where C is a certain constant. Therefore

$$F(b) - F(a) = F[\varphi(d)] - F[\varphi(c)]$$
$$= \Phi(d) - \Phi(c). \tag{11}$$

But on the basis of the Newton-Leibniz formula, the left-hand side of (11) is equal to the left-hand side of (10), and the right-hand side of (11) to the right-hand side of (10), and this proves formula (10).

Example 4.

$$\int_0^a \sqrt{a^2-x^2}\,dx = (x = a\sin t) = \int_0^{\pi/2} \sqrt{a^2-a^2\sin^2 t}\; a\cos t\,dt$$

$$= a^2 \int_0^{\pi/2} \cos^2 t\,dt = a^2 \int_0^{\pi/2} \frac{t+\cos 2t}{2}\,dt$$

$$= \frac{a^2}{2}\left[t+\frac{\sin 2t}{2}\right]_{t=0}^{t=\frac{\pi}{2}} = \frac{\pi a^2}{4}.$$

Remark. The upper limit of integration with respect to t can be taken equal to $\frac{5}{2}\pi$ and the result will be the same.

Example 5.

$$\int_0^{4\pi} \sin^3 t\,dt = -\int_0^{4\pi} (1-\cos^2 t)\,d(\cos t) = (x = \cos t)$$

$$= -\int_1^1 (1-x^2)\,dx = 0,$$

since in the obtained integral the lower limit is equal to the upper limit.

Example 6. If f is an even function ($f(-u)=f(u)$), then

$$\int_{-a}^{a} f(u)\,du = 2\int_0^a f(u)\,du,$$

since

$$\int_{-a}^{0} f(u)\,du = (u=-x) = -\int_a^0 f(-x)\,dx = \int_0^a f(-x)\,dx$$

$$= \int_0^a f(x)\,dx = \int_0^a f(u)\,du.$$

Example 7. If f is an odd function ($f(-u) = -f(u)$), then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Example 8. If f is a periodic function of period 2π ($f(x+2\pi) = f(x)$), then

$$\int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \int_{0}^{2\pi} f(x)\,dx,$$

since

$$\int_{2\pi}^{2\pi+\alpha} f(x)\,dx = (x = t + 2\pi) = \int_{0}^{\alpha} f(t+2\pi)\,dt$$

$$= \int_{0}^{\alpha} f(t)\,dt = -\int_{\alpha}^{0} f(t)\,dt,$$

and, consequently,

$$\int_{\alpha}^{2\pi+\alpha} f(x)\,dx = \int_{\alpha}^{0} f(x)\,dx + \int_{0}^{2\pi} f(x)\,dx + \int_{2\pi}^{2\pi+\alpha} f(x)\,dx$$

$$= \int_{0}^{2\pi} f(x)\,dx,$$

Example 9.

$$\int_{0}^{3\pi} \sin^3 t\,dt = (x = \cos t) = -\int_{1}^{-1} (1-x^2)\,dx = \int_{-1}^{1} (1-x^2)\,dx$$

$$= 2\int_{0}^{1} (1-x^2)\,dx = 2\left(x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{4}{3}.$$

Example 10. Solve Example 5, using Examples 8 and 7:

$$\int_{0}^{4\pi} \sin^3 t\,dt = \int_{0}^{2\pi} \sin^3 t\,dt + \int_{2\pi}^{4\pi} \sin^3 t\,dt = 2\int_{-\pi}^{\pi} \sin^3 t\,dt = 0,$$

since the function $\sin^3 t$ is odd.

Theorem 2. *The following formula of integration by parts takes place*

$$\int_a^b u'(x)\,v(x)\,dx = u(x)\,v(x)\Big|_a^b - \int_a^b u(x)\,v'(x)\,dx, \qquad (12)$$

where u and v are functions continuously differentiable on $[a, b]$.

Proof. The product $u(x)\,v(x)$ has on $[a, b]$ a continuous derivative

$$(u(x)\,v(x))' = u(x)\,v'(x) + u'(x)\,v(x).$$

Therefore, by the Newton-Leibniz theorem,

$$u(x)\,v(x)\Big|_a^b = \int_a^b [u(x)\,v'(x) + u'(x)\,v(x)]\,dx$$

$$= \int_a^b u(x)\,v'(x)\,dx + \int_a^b u'(x)\,v(x)\,dx,$$

whence follows (12).

Example 11.

$$\int_0^1 \ln(1+x)\,dx = (u = \ln(1+x),\ dv = dx)$$

$$= x\ln(1+x)\,\Big|_0^1 - \int_0^1 \frac{x\,dx}{1+x} = \ln 2 - \int_0^1 1\,dx + \int_0^1 \frac{dx}{x+1}$$

$$= \ln 2 - 1 + \ln(1+x)\,\Big|_0^1 = -1 + 2\ln 2.$$

Theorem 3 (on Mean Value for the Definite Integral). *For a function f continuous on a closed interval $[a, b]$ there exists a point $\xi \in (a, b)$ such that*

$$\int_a^b f(x)\,dx = f(\xi)\,(b-a). \qquad (13)$$

Proof. Since f is continuous, there exists an antiderivative for it, therefore

$$\int_a^b f(x)\,dx = \Phi(b) - \Phi(a) = \Phi'(\xi)(b-a) = f(\xi)(b-a),$$
$$\xi \in (a, b). \quad (14)$$

The first equality in (14) is the Newton-Leibniz formula for a function f continuous on $[a, b]$. The second equality is the Lagrange formula for Φ. Finally, the third equality follows from the fact that $\Phi'(x) = f(x)$, $\forall x \in [a, b]$.

Sec. 6.5. Taylor's Formula with the Remainder in the Integral Form

Let a function $f(x)$ have continuous derivatives up to order $n+1$ (included). Then, by the Newton-Leibniz formula, we have

$$f(x) = f(a) + \int_a^x f'(t)\,dt = \left(\begin{array}{l|l} u = f'(t) & du = f''(t)\,dt \\ dv = dt & v = t - x \end{array}\right)$$

$$= f(a) + (t-x) f'(t) \Big|_{t=a}^{t=x} - \int_a^x (t-x) f''(t)\,dt$$

$$= f(a) + f'(a)(x-a) + \int_a^x (x-t) f''(t)\,dt$$

$$= \left(\begin{array}{l|l} u = f''(t) & du = f'''(t)\,dt \\ (x-t)\,dt = dv & v = -\dfrac{(x-t)^2}{2!} \end{array}\right)$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \int_a^x f'''(t)\frac{(x-t)^2}{2!}\,dt.$$

Continuing the process of integration by parts, we obtain

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + r_n(x), \quad (1)$$

where

$$r_n(x) = \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t)\,dt. \tag{2}$$

Formula (1), (2) is called *Taylor's formula with the remainder in the integral form.*

Applying to integral (2) (with respect to t) Theorem 3 from the preceding section, we shall have

$$r_n(x) = \frac{1}{n!}(x-\xi)^n f^{(n+1)}(\xi)(x-a), \quad \xi \in (a, x).$$

Making

$$\xi = a + \theta(x-a), \quad 0 < \theta < 1,$$

we get

$$r(x) = \frac{(x-a)^{n+1}}{n!}(1-\theta)^n f^{(n+1)}(a+\theta(x-a)),$$

i.e. the remainder term of Taylor's formula in powers of $x-a$ in Cauchy's form (see Sec. 4.14, (10)).

Sec. 6.6. Darboux* Sums. Existence Conditions for the Integral

Let a bounded function f ($|f(x)| \leqslant M$) be given on a closed interval $[a, b]$ and let us introduce an arbitrary partition

$$R\colon a = x_0 < x_1 < \ldots < x_n = b.$$

Let

$$m_i = \inf_{x \in [x_i, x_{i+1}]} f(x), \quad M_i = \sup_{x \in [x_i, x_{i+1}]} f(x).$$

Along with the integral sums

$$\sigma_R = \sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i,$$

consider the sums

$$s_R = \sum_{i=0}^{n-1} m_i\,\Delta x_i, \quad S_R = \sum_{i=0}^{n-1} M_i\,\Delta x_i,$$

which are called the *lower and the upper Darboux sums*, respectively. It is obvious that $s_R \leqslant S_R$.

* Darboux, Jean Gaston (1842-1917). French mathematician.

Darboux sums are not necessarily integral sums. But if $f(x)$ is a continuous function, then s_R and S_R are, respectively, the least and the greatest of the integral sums corresponding to the given partition, since, according to the Weierstrass theorem, $f(x)$ attains a minimum and a maximum in each $[x_i, x_{i+1}]$ and therefore we can choose the points $\xi_i, \xi_i' \in [x_i, x_{i+1}]$ so that $f(\xi_i) = m_i$ and $f_i(\xi_i') = M_i$.

Since $m_i \leqslant f(x) \leqslant M_i$ and $\Delta x_i > 0$, we have

$$s_R \leqslant \sigma_R \leqslant S_R. \tag{1}$$

At a fixed partition s_R and S_R are constant numbers, and the integral sum σ_R remains variable by virtue of arbitrariness of the numbers ξ_i. It is easy to see that, by appropriately choosing the points ξ_i, the sum σ_R can be made arbitrarily close to s_R and S_R, i.e. for the given partition, s_R and S_R are, respectively, the infimum and the supremum for the integral sums:

$$s_R = \sum_{i=0}^{n-1} m_i \Delta x_i = \inf_{\xi_i} \sum_{i=0}^{n-1} f(\xi_i) \Delta x_i, \quad S_R = \sum_{i=0}^{n-1} M_i \Delta x_i = \sup_{\xi_i} \sum_{i=0}^{n-1} f(\xi_i) \Delta x_i.$$

Let R_1, R_2, R_3 be some partitions of $[a, b]$. If all points of division entering into R_1 belong to the system of points of division with the aid of which R_2 is formed we shall say that R_2 is a refinement of R_1 and shall write $R_1 \subset R_2$. If the set of points of division of R_3 is the set-theoretic sum of the points of division entering into R_1 and R_2, then we shall write $R_3 = R_1 + R_2$.

Considered below are the properties of Darboux sums.

1°. *If new points are added to the points of division of a partition R, then the upper Darboux sum (S_R) does not increase and the lower Darboux sum (s_R) does not decrease:*

$$S_{R'} \leqslant S_R, \quad s_R \leqslant s_{R'}, \quad \forall R \subset R'.$$

Hence,

$$S_{R'} - s_{R'} \leqslant S_R - s_R.$$

Proof. In proving this property, we may, obviously, confine ourselves to the case when only one new point of division $x' \in (x_i, x_{i+1})$ is added. Let S_R be the upper Darboux sum corresponding to the partition R and $S_{R'}$ corresponding to the partition R'. Then S_R differs from $S_{R'}$ by that, instead of the term $M_i \Delta x_i$, the sum $S_{R'}$ will contain two terms:

$$M_i' (x' - x_i) + M_i'' (x_{i+1} - x'),$$

where $M_i' = \sup\limits_{x \in [x_i, x']} f(x)$, $M_i'' = \sup\limits_{x \in [x', x_{i+1}]} f(x)$.

Since $[x_i, x']$, $[x', x_{i+1}]$ are parts of $[x_i, x_{i+1}]$, $M_i' \leqslant M_i$, $M_i'' \leqslant M_i$ (when reducing the domain of consideration, sup can only decrease). Therefore,

$$M_i'(x'-x_i)+M_i''(x_{i+1}-x') \leqslant M_i(x'-x_i+x_{i+1}-x')$$
$$= M_i(x_{i+1}-x_i),$$

i.e. $S_{R'} \leqslant S_R$, which was to be proved.

For lower sums the property is proved in a similar way.

2°. *Every lower Darboux sum is not greater than every upper Darboux sum, even if the latter corresponds to another partition of the interval:* $s_{R_1} \leqslant S_{R_2}$.

Proof. Let $R_3 = R_1 + R_2$. Taking into account Property 1°, we obtain $s_{R_1} \leqslant s_{R_3} \leqslant S_{R_3} \leqslant S_{R_2}$.

Thus, we have proved that the set of lower Darboux sums $\{s_R\}$ is bounded above by some upper sum $S_{R'}$ $(s_R \leqslant S_{R'})$, and therefore there *exists the supremum of lower sums*:

$$I_* = \sup_R s_R \leqslant S_{R'}.$$

Besides, we have proved that any upper sum $S_{R'}$ is not less than the number I_*. This shows that there *exists the infimum of upper sums*

$$I^* = \inf_{R'} S_{R'} \geqslant I_*.$$

Hence, $I_* \leqslant I^*$. And for any partition R the following inequalities are satisfied:

$$s_R \leqslant I_* \leqslant I^* \leqslant S_R. \tag{2}$$

The numbers I_*, I^* are called the *lower and the upper Darboux integrals*, respectively.

Theorem 1 (on the Existence of the Integral). *For the definite integral of a bounded function $f(x)$ to exist, it is necessary and sufficient that*

$$\lim_{\lambda_R \to 0} (S_R - s_R) = \lim_{\lambda_R \to 0} \sum_{i=0}^{n-1} \omega_i \Delta x_i = 0, \tag{3}$$

where the number $\omega_i = M_i - m_i$ is called the oscillation of the function $f(x)$ on $[x_i, x_{i+1}]$.

Proof 1. The Necessity of the Condition. Let us assume that the definite integral I of the function $f(x)$ exists: i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $I - \varepsilon < \sigma_R < I + \varepsilon$, as soon as $\lambda_R < \delta$ irrespective of the choice of the points $\xi_i \in [x_i, x_{i+1}]$.

As was established above, for a given R, s_R and S_R are the infimum and the supremum, respectively, for the integral sums σ_R, provided the points $\xi_i \in [x_i, x_{i+1}]$ are varied. Therefore,

$$I - \varepsilon \leqslant s_R \leqslant \sigma_R \leqslant S_R \leqslant I + \varepsilon, \quad \forall R \text{ with } \lambda_R < \delta,$$

i. e.

$$\lim_{\lambda_R \to 0} s_R = I, \quad \lim_{\lambda_R \to 0} S_R = I$$

and

$$\lim_{\lambda_R \to 0} (S_R - s_R) = 0.$$

2. The Sufficiency of the Condition. Let condition (3) be fulfilled. Then, it follows from inequality (2) that $I_* = I^*$. Let us denote the common value of these two numbers by I ($I_* = I^* = I$). Then

$$s_R \leqslant I \leqslant S_R. \tag{4}$$

It follows from (3) that for any $\varepsilon > 0$, $\exists \delta > 0$, such that $|S_R - s_R| < \varepsilon$ for $\lambda_R < \delta$. But then from (1) and (4) we get

$$|I - \sigma_R| < \varepsilon \quad \text{for} \quad \lambda_R < \delta,$$

i.e. I is a limit for σ_R and $f(x)$ is integrable.

Remark. From the proof of the above theorem it is seen that if a function $f(x)$ is integrable on $[a, b]$, then $\lim_{\lambda_R \to 0} s_R =$ $= \lim_{\lambda_R \to 0} S_R = \int_a^b f(x)\,dx$, and, conversely, if $\lim_{\lambda_R \to 0} s_R = \lim_{\lambda_R \to 0} S_R = I$, then $I = \int_a^b f(x)\,dx$.

Sec. 6.7. Integrability of Continuous and Monotone Functions

Theorem 1. *If a function $f(x)$ is continuous on a closed interval $[a, b]$, then it is integrable on that interval.*

Proof. Since the function $f(x)$ is continuous on $[a, b]$, then it is uniformly continuous on $[a, b]$ and, consequently, $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that, as soon as $[a, b]$ is partitioned into subintervals with $\lambda_R < \delta$, all oscillations $\omega_i < \varepsilon$. Hence,

$$\sum_{i=0}^{n-1} \omega_i \,\Delta x_i \leqslant \varepsilon \sum_{i=0}^{n-1} \Delta x_i = \varepsilon (b - a).$$

By virtue of the arbitrariness of ε, we conclude that $\lim_{\lambda_R \to 0} \sum_{i=0}^{n-1} \omega_i \,\Delta x_i = 0$ and, by Theorem 1 proved in Sec. 6.6, the function $f(x)$ is integrable.

Theorem 2. *A function $f(x)$ monotone on a closed interval $[a, b]$ is integrable on this interval.*

Proof. For the sake of definiteness, we shall assume that $f(x)$ is not decreasing. We shall also assume that $f(a) < f(b)$, otherwise the function is constant and the theorem is trivial.

Since $f(a) \leqslant f(x) \leqslant f(b)$, $\forall x \in [a, b]$, our function is bounded on $[a, b]$. Let us introduce a partition R of the interval $[a, b]$ with $\lambda_R < \delta$. Since in the present case $\omega_i = f(x_{i+1}) - f(x_i)$, we have

$$\sum_{i=0}^{n-1} \omega_i \Delta x_i \leqslant \delta \sum_{i=0}^{n-1} \omega_i = \delta [f(x_1) - f(x_0) + f(x_2) - f(x_1) + \ldots + f(x_n)$$
$$- f(x_{n-1})] = \delta [f(b) - f(a)],$$

$x_0 = a$, $x_n = b$. Let us now choose $\delta = \varepsilon/[f(b) - f(a)]$, then

$$\sum_{i=0}^{n-1} \omega_i \Delta x_i \leqslant \varepsilon,$$

and, applying the existence theorem (Theorem 1 in Sec. 6.6), we conclude that $f(x)$ is integrable. The theorem has been proved.

Remark 1. Note that a monotone function can have a countable set of points of discontinuity. For instance, the function $y = \left\{x + \frac{1}{n}, \frac{1}{n+1} < x \leqslant \frac{1}{n}, n = 1, 2, \ldots\right\}$ increases monotonically on $[0, 1]$ and has a countable set of points of discontinuity. Consequently, by Theorem 2, it is integrable.

Remark 2. If $f(x)$ is integrable on $[a, b]$ $(a < b)$, then $|f(x)|$ is also integrable.

Indeed, $\forall x'$ and x'' from $[x_i, x_{i+1}]$ we have

$$\left| |f(x')| - |f(x'')| \right| \leqslant |f(x') - f(x'')|. \tag{1}$$

If ω_i^*, ω_i are respective oscillations of $|f(x)|$, and $f(x)$, on $[x_i, x_{i+1}]$, then it follows from (1) that $\omega_i^* \leqslant \omega_i$ and

$$\sum_{i=0}^{n-1} \omega_i^* \Delta x_i \leqslant \sum_{i=0}^{n-1} \omega_i \Delta x_i.$$

Since $f(x)$ is integrable, we have

$$\sum_{i=0}^{n-1} \omega_i \Delta x_i \to 0 \text{ for } \lambda_R \to 0,$$

but then

$$\sum_{i=0}^{n-1} \omega_i^* \Delta x_i \to 0,$$

and, consequently, $|f(x)|$ is integrable.

Sec. 6.8. Improper Integrals

Let a function f be defined in a half-open finite interval $[a, b)$. We shall suppose that it is integrable (for instance, continuous or piecewise continuous) on any closed interval $[a, b']$ with $b' < b$ and is unbounded in a neighbourhood of the point b. Then the ordinary (Riemann) integral of f on $[a, b)$ or, which is the same, on $[a, b]$ must be bounded. But it may happen that the finite limit

$$\lim_{b' \to b} \int_a^{b'} f(x)\, dx,$$

exists. If it does exist we call it the *improper integral of* f on the interval $[a, b]$ and write

$$\int_a^b f(x)\, dx = \lim_{b' \to b} \int_a^{b'} f(x)\, dx. \tag{1}$$

In such a case we say that the (improper) *integral* $\int_a^b f\, dx$ *exists* or that it is *convergent*. If otherwise, we say that the integral is *divergent* or that it does not exist (in the sense of an improper Riemann integral).

Now let us suppose that a function f is defined in a semifinite interval (ray) $[a, \infty)$ and is integrable on every finite closed interval $[a, b']$, where $a < b' < \infty$. Then if the limit

$$\lim_{b' \to \infty} \int_a^{b'} f(x)\, dx,$$

exists, we call it the *improper integral of* f *on* $[a, \infty)$ and write

$$\int_a^\infty f(x)\, dx = \lim_{b' \to \infty} \int_a^{b'} f(x)\, dx.$$

We shall use the following terminology. The expression

$$\int_a^b f(x)\, dx \tag{2}$$

will be referred to as the *integral (of f) with only one singularity at the point b* if the following conditions hold: if b is a finite point, then the function f is integrable on $[a, b']$ for any b' satisfying the inequalities $a < b' < b$ and is unbounded in a neighbourhood of the point b. If $b = +\infty$, then it is only required that the function f should be integrable on $[a, b']$ for any finite $b' > a$.

An integral of the form $\int_a^b f(x)\,dx$ with only one singularity at the point a is understood in the same sense. In this case b is a finite point, and if the point $a < b$ is also finite, then f is unbounded in a neighbourhood of the point a and is integrable on every closed interval $[a', b]$, where $a < a' < b$. If $a = -\infty$, then the function f is supposed to be integrable on $[a', b]$ for any $a' < b$.

For the sake of definiteness, we shall consider integral (2) with only one singularity at the point b, which can be finite or infinite. By analogy with this case, all the conclusions can readily be extended to the case of integral (2) with only one singularity at the point a.

Theorem. *Let there be given an integral of type* (2) *with only one singularity at the point b. For the given integral to exist, it is necessary and sufficient that the conditions of Cauchy's criterion hold: given any $\varepsilon > 0$, there is $b_0 < b$ such that*

$$\left|\int_{b'}^{b''} f(t)\,dt\right| < \varepsilon, \tag{3}$$

for any b' and b'' satisfying the inequalities $b_0 < b' < b'' < b$.

Proof. Let us consider the function

$$F(x) = \int_a^x f(t)\,dt \quad (a < x < b).$$

The existence of integral (2) is equivalent to the existence of the limit $\lim\limits_{\substack{x \to b \\ x < b}} F(x)$, which, in its turn, is equivalent

to the fulfilment of Cauchy's condition: for any $\varepsilon > 0$ there is b_0, $a < b_0 < b$, such that for all b' and b'', satisfying the inequalities $b_0 < b' < b'' < b$, the inequality $|F(b'') - F(b')| < \varepsilon$ holds. But

$$F(b'') - F(b') = \int_{b'}^{b''} f(t)\,dt,$$

and the theorem has been proved.

Example 1. The integral

$$\int_0^1 \frac{dx}{x^{\alpha}}, \tag{4}$$

where $\alpha > 0$ is a constant number, has, obviously, only one singularity at the point $x = 0$. To find out whether the integral is convergent, we must compute the limit

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{\varepsilon}^{1} \frac{dx}{x^{\alpha}} = \lim_{\varepsilon\to 0} \frac{x^{1-\alpha}}{1-\alpha}\bigg|_{\varepsilon}^{1} = \lim_{\varepsilon\to 0} \frac{1}{1-\alpha}[1-\varepsilon^{1-\alpha}]$$

$$= \begin{cases} \dfrac{1}{1-\alpha}, & \alpha < 1, \\ \infty, & \alpha > 1. \end{cases}$$

Thus, integral (4) is convergent for $\alpha < 1$ and is equal to $(1-\alpha)^{-1}$ in this case and is divergent for $\alpha > 1$. For $\alpha = 1$ it is also divergent

$$\lim_{\varepsilon\to 0} \int_{\varepsilon}^{1} \frac{dx}{x} = -\lim_{\varepsilon\to 0} \ln \varepsilon = +\infty.$$

Example 2. The integral

$$\int_1^{\infty} \frac{dx}{x^{\alpha}} = \lim_{N\to\infty} \int_1^N \frac{dx}{x^{\alpha}} = \frac{1}{1-\alpha} \lim_{N\to\infty} x^{1-\alpha}\big|_1^N$$

$$= \begin{cases} \dfrac{1}{\alpha-1}, & \alpha > 1 \text{ (converges)} \\ +\infty, & \alpha < 1 \text{ (diverges)}, \end{cases}$$

$$\int_1^{\infty} \frac{dx}{x} = \lim_{N\to\infty} \int_1^N \frac{dx}{x} = \lim_{N\to\infty} \ln N = +\infty \text{ (diverges)}.$$

Example 3. The integral $\int\limits_0^{\infty} e^{-x}\, dx$ has only one singularity at the point $x = +\infty$. It is convergent and is equal to

$$\int\limits_0^{\infty} e^{-x}\, dx = \lim_{N\to+\infty} \int\limits_0^{N} e^{-x}\, dx = \lim_{N\to+\infty} (-e^{-x}) \Big|_0^N$$
$$= \lim_{N\to+\infty} [1 - e^{-N}] = 1.$$

Let us come back to the general expression of an integral

$$\int\limits_a^b f(x)\, dx, \tag{5}$$

having only one singularity at the point b. Then the integral

$$\int\limits_c^b f(x)\, dx, \tag{6}$$

where $a < c < b$ also has only one singularity at the point b. It is obvious that the conditions of Cauchy's criterion for the existence of the integral are stated in exactly the same manner for both integrals (5) and (6). Consequently, these integrals are simultaneously convergent or divergent. For $a < c < b$ we also have the obvious relation

$$\int\limits_a^b f\, dx = \lim_{b'\to b} \int\limits_a^{b'} f\, dx = \lim_{b'\to b} \left(\int\limits_a^{c} f\, dx + \int\limits_c^{b'} f\, dx \right)$$
$$= \int\limits_a^{c} f\, dx + \lim_{b'\to b} \int\limits_c^{b'} f\, dx = \int\limits_a^{c} f\, dx + \int\limits_c^{b} f\, dx, \tag{7}$$

where $\int\limits_a^c$ is an ordinary (proper) Riemann integral, while the integrals $\int\limits_a^b$ and $\int\limits_c^b$ are improper.

We also note that for any constants A and B the equality

$$\int_a^b (Af+B\varphi)\,dx=\lim_{b'\to b}\int_a^{b'}(Af+B\varphi)\,dx$$

$$=A\lim_{b'\to b}\int_a^{b'} f\,dx+B\lim_{b'\to b}\int_a^{b'}\varphi\,dx=A\int_a^b f\,dx+B\int_a^b \varphi\,dx \qquad (8)$$

takes place. It should be understood in the sense that if the integrals on the right-hand side exist, then the integral on the left-hand side also exists and equality (8) holds.

An integral of type (5) (with only one singularity at the point b) is said to be *absolutely convergent* if the integral

$$\int_a^b |f(x)|\,dx \qquad (9)$$

of the modulus of the function $f(x)$ is convergent.

Let us prove that an *absolutely convergent integral is convergent.* Suppose that integral (9) is convergent. Then, given any $\varepsilon>0$, there is a point b_0 belonging to the open interval (a, b) such that the inequalities $b_0<$ $<b'<b''<b$ imply

$$\varepsilon>\int_{b'}^{b''}|f(x)|\,dx\geqslant\left|\int_{b'}^{b''}f(x)\,dx\right|,$$

that is, integral (5) satisfies Cauchy's condition. Since

$$\left|\int_a^{b'}f(x)\,dx\right|\leqslant\int_a^{b'}|f(x)|\,dx,$$

we conclude, on passing to the limit as $b'\to b$, that for an absolutely convergent integral of type (5) the inequality

$$\left|\int_a^b f(x)\,dx\right|\leqslant\int_a^b|f(x)|\,dx \qquad (10)$$

is always fulfilled.

Remark. Inequality (10) is also true for a not absolutely convergent integral. In this case on the right stands the symbol ∞ which is regarded to be greater than any finite number. This is widely used in the computing technique. If it is required to find out whether the integral $\int_a^b f\,dx$ is convergent or not, we write inequality (10) and investigate the integral $\int_a^b |f|\,dx$ for convergence. If the latter converges, i.e. if $\int_a^b |f|\,dx < \infty$, then our integral $\int_a^b f\,dx$ also converges. Of course, if $\int_a^b |f|\,dx = \infty$, more accurate methods should be applied to our integral. Possibly, it is still convergent, but not absolutely (see examples at the end of the next section).

Sec. 6.9. Improper Integrals of Nonnegative Functions

Let there be given the integral

$$\int_a^b f(x)\,dx, \tag{1}$$

having only one singularity at the point b, and let $f(x) \geqslant 0$ in the interval of integration $[a, b)$. Then, obviously, the function

$$F(b') = \int_a^{b'} f(x)\,dx \quad (a < b' < b)$$

of b' is not monotonically decreasing. Therefore, if it is bounded $(F(b') \leqslant M,\ a < b' < b)$, then there exists integral (1)

$$\int_a^b f(x)\,dx = \lim_{b' \to b} \int_a^{b'} f(x)\,dx \leqslant M.$$

And if F is unbounded, then integral (1) is divergent:

$$\int_a^b f(x)\,dx = \lim_{b' \to b} \int_a^{b'} f(x)\,dx = +\infty.$$

If $f(x) \geqslant 0$ on $[a, b)$, we write

$$\int_a^b f(x)\,dx < \infty \quad \text{or} \quad \int_a^b f(x)\,dx = \infty$$

depending on whether the integral will converge or diverge.

Theorem 1. *Let each of the integrals*

$$\int_a^b f(x)\,dx, \tag{1}$$

$$\int_a^b \varphi(x)\,dx, \tag{2}$$

have only one singularity at the point b and let the inequalities

$$0 \leqslant f(x) \leqslant \varphi(x) \tag{3}$$

be fulfilled in the interval $[a, b)$.

Then the convergence of integral (2) *implies the convergence of integral* (1) *and the inequality*

$$\int_a^b f\,dx \leqslant \int_a^b \varphi\,dx,$$

takes place, while the divergence of integral (1) *implies the divergence of integral* (2).

Proof. Inequalities (3) imply that

$$\int_a^{b'} f\,dx \leqslant \int_a^{b'} \varphi\,dx \tag{4}$$

for $a < b' < b$. If integral (2) is convergent, then the right-hand member of (4) is bounded by a number equal to integral (2), but then the left-hand member is also

bounded above. Since the left member is monotonically nondecreasing as b' increases, it tends to a limit which is equal to the integral:

$$\int_a^b f\,dx = \lim_{b'\to b} \int_a^{b'} f\,dx \leqslant \int_a^b \varphi\,dx.$$

Now, if integral (1) is divergent, then the limit of the left member of (4) as $b' \to b$ is equal to ∞, and, consequently, the limit of the right member is also equal to ∞.

Theorem 2. *Let each of the integrals* (1) *and* (2) *have only one singularity at the point* b, *let the integrands be positive and the limit*

$$\lim_{x\to b} \frac{f(x)}{\varphi(x)} = A > 0. \qquad (5)$$

exist. Then the integrals are simultaneously convergent or divergent.

Proof. It follows from (5) that for any positive $\varepsilon < A$ there is $c \in [a, b)$, such that

$$A - \varepsilon < \frac{f(x)}{\varphi(x)} < A + \varepsilon \quad (c < x < b),$$

and since $\varphi(x) > 0$, we obtain

$$(A - \varepsilon)\,\varphi(x) < f(x) < (A + \varepsilon)\,\varphi(x) \quad (c < x < b). \qquad (6)$$

The convergence of the integral $\int_a^b \varphi\,dx$ implies the convergence of the integral $\int_c^b \varphi\,dx$ and also the convergence of the integral $\int_c^b (A + \varepsilon)\,\varphi\,dx$. Therefore, by virtue of the foregoing theorem, the integral $\int_c^b f\,dx$

also converges and, together with it, so does the integral $\int_a^b f\,dx$. Conversely, if $\int_a^b f\,dx$ is convergent, then $\int_c^b \varphi\,dx$ is also convergent, since, along with (5), there holds the equality

$$\lim_{x\to b}\frac{\varphi(x)}{f(x)}=\frac{1}{A}>0.$$

Remark. The equality (5) means that the function f is equivalent to the function $A\varphi$ for $x\to b$. In this case the functions f and φ are said to have the same order for $x\to b$.

Example 1. Investigate the given integral for convergence:

$$\int_0^\infty \sin kxe^{-x}\,dx.$$

We have

$$\left|\int_0^\infty e^{-x}\sin kx\,dx\right|\leqslant\int_0^\infty |e^{-x}\sin kx|\,dx\leqslant\int_0^\infty e^{-x}\,dx=1<\infty.$$

We have used inequality (10) from Sec. 6.8 and the relevant remark.

Let us agree that the symbol $\sim$ placed between two integrals means that they are simultaneously convergent or divergent (by virtue of Theorem 2).

Example 2. $\int_0^1 \frac{dx}{\sin x}\sim\int_0^1\frac{dx}{x}=\infty.$

Example 3. $\int_0^1 \frac{dx}{\sin\sqrt{x}}\sim\int_0^1\frac{dx}{\sqrt{x}}<\infty.$

Example 4. $\int_1^\infty \frac{x-1}{x}e^{-x}\,dx\sim\int_1^\infty e^{-x}\,dx<\infty.$

The singularities in the integrals of Examples 2 and 3 are only at the point $x = 0$. It should be taken into account that $\sin x \approx x$, and $\sin \sqrt{x} \approx \sqrt{x}$, $x \to 0$.

The integral in Example 4 has only one singularity at $x = \infty$. It should be borne in mind that $\frac{x-1}{x} e^{-x} \approx$ $\approx e^{-x}$, $x \to +\infty$.

Example 5. $\int\limits_0^\infty (x^2 - 3x + 5)\, e^{-x}\, dx$ is convergent, since

$$\left| \int\limits_0^\infty (x^2 - 3x + 5)\, e^{-x}\, dx \right| \leqslant \int\limits_0^\infty \left| (x^2 - 3x + 5)\, e^{-x/2} \right| e^{-x/2}\, dx$$

$$\leqslant M \int\limits_0^\infty e^{-x/2}\, dx < \infty.$$

The point is that $\lim\limits_{x\to\infty} (x^2 - 3x + 5)\, e^{-x/2} = 0$, therefore there is $N > 0$ such that $| (x^2 - 3x + 5)\, e^{-x/2} | < 1$, $\forall x > N$.

On the other hand, the function $| (x^2 - 3x + 5)\, e^{-x/2} |$ is continuous on $[0, N]$ and, consequently, bounded on $[0, N]$ by a certain number M_1. Hence, it is bounded on $[0, \infty)$ by the number $M = \max \{1, M_1\}$.

Sec. 6.10. Improper Integrals Integrated by Parts

Example 1. The improper integrals

$$\int\limits_a^\infty \frac{\sin x}{x}\, dx, \quad \int\limits_a^\infty \frac{\cos x}{x}\, dx \quad (a > 0) \tag{1}$$

are convergent. Indeed, integrating by parts, we obtain

$$\int\limits_a^A \frac{\sin x}{x}\, dx = -\frac{\cos x}{x}\Bigg|_a^A - \int\limits_a^A \frac{\cos x}{x^2}\, dx$$

for all finite $A > a$. Passing now to the limit as $A \to \infty$, we get

$$\int_a^\infty \frac{\sin x}{x}\,dx = \frac{\cos a}{a} - \int_a^\infty \frac{\cos x}{x^2}\,dx,$$

where the integral in the right-hand side converges and even absolutely:

$$\left|\int_a^\infty \frac{\cos x}{x^2}\,dx\right| \leqslant \int_a^\infty \frac{|\cos x|}{x^2}\,dx \leqslant \int_a^\infty \frac{dx}{x^2} = \frac{1}{a} < \infty.$$

Example 2. The integral $\int_a^\infty \frac{\sin x}{x}\,dx$ converges not absolutely (i.e. conditionally), since the integral

$$\int_a^\infty \frac{|\sin x|}{x}\,dx = \infty, \quad a > 0, \tag{2}$$

i.e. divergent. Indeed, by virtue of the inequality $\sin^2 x \leqslant \leqslant |\sin x|$, the improper integral

$$\int_a^\infty \frac{|\sin x|}{x}\,dx \geqslant \int_a^\infty \frac{\sin^2 x}{x}\,dx = \int_a^\infty \frac{1-\cos 2x}{2x}\,dx$$

$$= (2x = u) = \frac{1}{2}\int_{2a}^\infty \frac{1-\cos u}{u}\,du = \frac{1}{2}\int_{2a}^\infty \frac{du}{u} - \frac{1}{2}\int_{2a}^\infty \frac{\cos u}{u}\,du.$$

But the integral $\int_{2a}^\infty u^{-1}\cos u\,du$ is convergent, while the integral $\int_{2a}^\infty u^{-1}\,du$ is divergent. Consequently, improper integral (2) is divergent.

Remark. The convergence of integral (1) is explained by the fact that the function $\sin x$ periodically oscillates, assuming, in succession, positive and negative values. The accumulation of area caused by the positive values

of sin x is compensated by the corresponding accumulation caused by the negative values.

This phenomenon will be explained when studying the theory of series (see Leibniz's series and conditionally convergent series).

The above examples show that integration by parts may turn out to be a useful means of investigation of improper integrals for convergence.

Given below are general considerations presenting a better explanation of this method.

Let a function $\varphi(x)$ be continuous on $[a, \infty)$ and $\Phi(x)$ be its antiderivative. Let also $g(x)$ be a function continuously differentiable on $[a, \infty)$. Then

$$\int_a^A \varphi(x)\, g(x)\, dx = g(x)\,\Phi(x)\Big|_a^A - \int_a^A \Phi(x)\, g'(x)\, dx$$

$$= g(A)\,\Phi(A) - g(a)\,\Phi(a) - \int_a^A \Phi(x)\, g'(x)\, dx. \qquad (3)$$

If

(1) $\lim\limits_{A\to\infty} g(A)\,\Phi(A) = 0,$

and

(2) the integral $\int_a^\infty \Phi(x)\, g'(x)\, dx$ is convergent, then, obviously, there exists the improper integral

$$\int_a^\infty \varphi(x)\, g(x)\, dx = \lim_{A\to\infty} \int_a^A \varphi(x)\, g(x)\, dx = -g(a)\,\Phi(a)$$

$$- \int_a^\infty \Phi(x)\, g'(x)\, dx. \qquad (4)$$

Hence, in particular, there follows

Dirichlet's test for the convergence of integral (4). If the function $\Phi(x)$ is bounded ($\Phi(x) \leqslant M$), and $g(x)$ decreases and tends to zero as $x \to \infty$, then integral (4) is convergent.

It is clear that these conditions imply Property (1). Further

$$\left|\int_a^\infty \Phi(x)\, g'(x)\, dx\right| \leqslant \int_a^\infty |\Phi(x)\, g'(x)|\, dx \leqslant M \int_a^\infty |g'(x)|\, dx$$

$$= -M \int_a^\infty g'(x)\, dx = -M \lim_{A\to\infty} \int_a^A g'(x)\, dx = -M \lim_{A\to\infty} [g(A) - g(a)]$$

$$= g(a)\cdot M.$$

Example 3. The integral

$$\int_a^\infty \frac{\sin x}{x^\alpha}\, dx \qquad (\alpha > 0),$$

having only one singularity at $x = \infty$, is convergent for $\alpha > 0$. This follows from Dirichlet's test, where we set $g(x) = x^{-\alpha}$ and $\varphi(x) = \sin x$, $\Phi(x) = -\cos x$ $(|\Phi(x)| \leqslant 1)$. It is absolutely convergent only for $\alpha > 1$, which is proved as in Example 2 given in Sec. 6.9.

Sec. 6.11. Improper Integrals with Singularities at Several Points

Let there be given an integral

$$\int_a^b f(x)\, dx, \tag{1}$$

i.e. for the time being a formal expression, in which under the sign $\int_a^b$ stands a function $f(x)$ defined on an open interval (a, b). Hence, a is either a finite number, or $-\infty$, and b is a finite number or $+\infty$.

Suppose the interval (a, b) can be partitioned into a finite number of subintervals by means of points $a = c_0 < c_1 < c_2 < \ldots < c_N = b$ so that every integral

$$\int_{c_k}^{c_{k+1}} f(x)\, dx \quad (k = 0, 1, \ldots, N-1) \tag{2}$$

has only one singularity either at the point c_k, or at the point c_{k+1}.

If all improper integrals (2) are convergent (absolutely convergent), then integral (1) is called an *improper convergent* (*absolutely convergent*) integral and expression (symbol) (1) is equated to a number

$$\int_a^b f(x)\,dx = \sum_{k=0}^{N-1} \int_{c_k}^{c_{k+1}} f(x)\,dx.$$

But if at least one of integrals (2) is divergent, then integral (1) is regarded as divergent.

If $f(x) \geqslant 0$, then, the same as in the case of integrals with one singularity, for integral (1) we shall write

$$\int_a^b f(x)\,dx < \infty,$$

if it is convergent and

$$\int_a^b f(x)\,dx = \infty,$$

if it is divergent.

Example 1.

$$\int_{-\infty}^{\infty} e^{-x}\,dx = \int_{-\infty}^{0} e^{-x}\,dx + \int_0^{\infty} e^{-x}\,dx = \infty + 1 = \infty.$$

This integral has two singularities: at $x = -\infty$ and $x = \infty = +\infty$. Accordingly we represent it formally as a sum of two integrals, each of which has one of the indicated singularities. Obviously,

$$\int_{-\infty}^{0} e^{-x}\,dx = \infty, \quad \int_0^{\infty} e^{-x}\,dx = 1.$$

Here we have set $\infty \pm 1 = \infty$,

Example 2. $(\alpha > 0)$

$$\int_0^\infty \frac{\sin x}{x^\alpha}\,dx - \begin{cases} \text{converges conditionally for } 0<\alpha\leqslant 1, \\ \text{converges absolutely for } 1<\alpha<2, \\ \text{diverges for } \alpha\geqslant 2. \end{cases} \tag{3}$$

Indeed, this integral has two singularities: at $x = 0$ and $x = \infty$, therefore, in order to investigate it, let us consider the formal sum

$$\int_0^\infty \frac{\sin x}{x^\alpha}\,dx = \int_0^1 \frac{\sin x}{x^\alpha}\,dx + \int_1^\infty \frac{\sin x}{x^\alpha}\,dx.$$

Under the integral sign $\int_0^1$ stands a positive function, therefore this integral is either divergent or absolutely convergent. It can be investigated with the aid of the inequalities (see Sec. 3.3, (6) and Sec. 4.19, Example 1)

$$\frac{2}{\pi}x^{1-\alpha} \leqslant \frac{\sin x}{x^\alpha} \leqslant x^{1-\alpha} \quad (0<x\leqslant 1),$$

whence

$$\int_0^1 \frac{\sin x}{x^\alpha}\,dx \leqslant \int_0^1 x^{1-\alpha}\,dx < \infty \quad \text{for } \alpha<2,$$

$$\int_0^1 \frac{\sin x}{x^\alpha}\,dx \geqslant \frac{2}{\pi}\int_0^1 x^{1-\alpha}\,dx = \infty \quad \text{for } \alpha\geqslant 2.$$

Consequently,

$$\int_0^1 \frac{\sin x}{x^\alpha}\,dx - \begin{cases} \text{converges absolutely for } \alpha<2, \\ \text{diverges for } \alpha\geqslant 2. \end{cases} \tag{4}$$

Further (see Sec. 6.10, Example 2)

$$\int_1^\infty \frac{\sin x}{x^\alpha}\,dx - \begin{cases} \text{converges for } \alpha>0, \\ \text{converges absolutely only for } \alpha>1, \end{cases} \tag{5}$$

CHAPTER 7

SOME APPLICATIONS OF INTEGRALS. APPROXIMATE METHODS

Sec. 7.1. Area in Polar Coordinates

The area S of the figure bounded by two rays $\theta = \theta_0$, $\theta = \theta_*$ issued from the pole O and by a curve Γ represented in polar coordinates by a continuous function $\rho = f(\theta)$ can be computed in the following way (see Fig. 80).

We partition the interval $[\theta_0, \theta_*]$ of variation of θ with the aid of points of division

$$\theta_0 < \theta_1 \ldots < \theta_n = \theta_*$$

and take the expression

$$\frac{1}{2}\rho_k^2 \Delta\theta_k, \quad \Delta\theta_k = \theta_{k+1} - \theta_k$$

as an approximation to the area of the element of the figure bounded by the curve Γ and by two rays $\theta = \theta_k$,

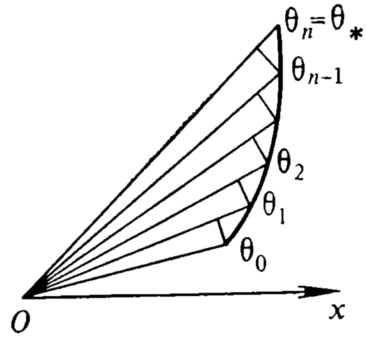

Fig. 80

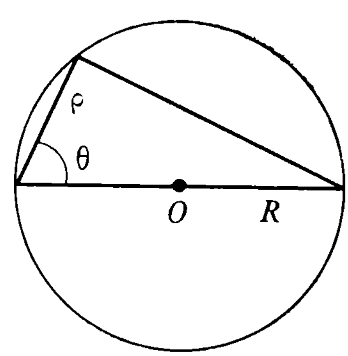

Fig. 81

$\theta = \theta_{k+1}$; this means that we replace the area of ths element by the area $k + 1$ of the circular sector of radius $\rho_k = f(\theta_k)$ bounded by the same rays. Now it appeare natural to define the area of the figure under consideration

by setting

$$S = \lim_{\max \Delta\theta_k \to 0} \frac{1}{2} \sum_{k=0}^{n-1} \rho_k^2 \, \Delta\theta_k = \frac{1}{2} \int_{\theta_0}^{\theta_*} \rho^2 \, d\theta = \frac{1}{2} \int_{\theta_0}^{\theta_*} f^2(\theta) \, d\theta. \quad (1)$$

Formula (1) expresses the area S of the figure in polar coordinates. As we know, integral (1) is sure to exist if the function $f(\theta)$ is continuous, and, consequently, the limit of any integral sum is equal to this integral.

Example. The circumference of the circle shown in Fig. 81 is described in polar coordinates by the equation $\rho = 2R \cos\theta$. By (1), its area

$$S = 2R^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta = 4R^2 \int_0^{\pi/2} \frac{1+\cos 2\theta}{2} \, d\theta = \pi R^2.$$

Sec. 7.2. Volume of a Solid of Revolution

Let Γ be a curve described in rectangular coordinates x, y by a continuous positive function $y = f(x)$ $(a \leqslant x \leqslant b)$. We shall compute the volume V of the solid of revolution bounded by the planes $x = a$ and $x = b$ and by the surface of revolution generated by the rotation of the curve Γ about the x-axis.

Let us partition the interval $[a, b]$ with the aid of points of division $a = x_0 < x_1 < \ldots < x_n = b$ and assume that the volume of the element ΔV of the solid bounded by the planes $x = x_k$ and $x = x_{k+1}$ is approximately equal to the volume of the circular cylinder with radius $y_k = f(x_k)$ and altitude $\Delta x_k = x_{k+1} - x_k$:

$$\Delta V_k \sim \pi y_k^2 \Delta x_k = \pi f(x_k)^2 \, \Delta x_k.$$

The quantity $V_n = \pi \sum_{k=0}^{n-1} f^2(x_k) \, \Delta x_k$ approximately expresses V and

$$V = \lim_{\max \Delta x_k \to 0} \pi \sum_{k=0}^{n-1} f^2(x_k) \, \Delta x_k = \pi \int_a^b f^2(x) \, dx. \quad (1)$$

We have obtained the formula for computing the volume of the solid of revolution (Fig. 82).

Example. The ellipsoid of revolution (about the x-axis)

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}\leqslant 1$$

is a body bounded by the surface of revolution generated

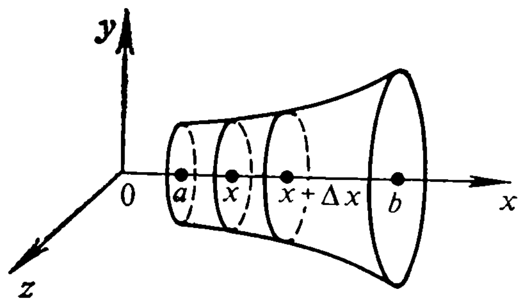

Fig. 82

by rotating the curve

$$y=b\sqrt{1-\frac{x^2}{a^2}} \quad (-a\leqslant x\leqslant a)$$

about the x-axis, therefore, by formula (1), its volume is equal to

$$V=\pi b^2\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)dx=\pi b^2\left(x-\frac{x^3}{3a^2}\right)\Big|_{-a}^{a}=\frac{4}{3}\pi ab^2.$$

Sec. 7.3. A Smooth Curve in Space. Arc Length

In Sec. 4.21 we introduced the notion of a plane continuous curve represented parametrically and of a smooth curve, in particular.

Here we want to add some more details and, at the same time, to consider a more general curve in space.

Three equations (Fig. 83)

$$\left.\begin{aligned} x&=\varphi(t),\\ y&=\psi(t),\\ z&=\chi(t),\end{aligned}\right\} \quad a\leqslant t\leqslant b, \tag{1}$$

where the functions φ, ψ, χ are continuous on a closed interval $[a, b]$, define a *continuous curve* which will be denoted by Γ. Furthermore, if these functions are not only continuous, but also have on $[a, b]$ continuous derivatives, not simultaneously vanishing, then Γ is called a *smooth curve*.

The fact that the derivatives $\varphi'(t)$, $\psi'(t)$, $\chi'(t)$ are not all zero for any $t \in [a, b]$ can be expressed as follows: the inequality

$$(\varphi'(t))^2 + (\psi'(t))^2 + (\chi'(t))^2 > 0 \qquad (2)$$

takes place for all $t \in [a, b]$.

If we take a definite value $t = t_0$, then, by (2) one of the terms $\varphi'(t_0)$, $\psi'(t_0)$, $\chi'(t_0)$, say the first, is not equal to zero ($\varphi'(t_0) \neq 0$). Since φ' is continuous, there

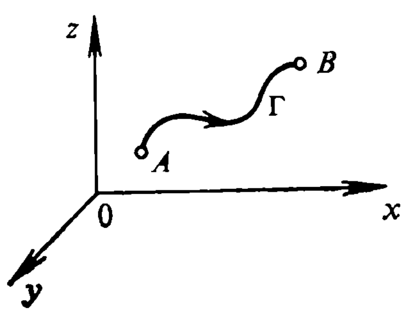

Fig. 83

exists an interval $(t_0 - \delta, t_0 + \delta)$ on which $\varphi'(t)$ has the same sign as $\varphi'(t_0)$. But then the function $x = \varphi(t)$ is strictly monotone on that interval and there exists its inverse, a continuously differentiable function $t = \varphi^{-1}(x)$, $x \in (c, d)$, where (c, d) is a certain neighbourhood of the point $x_0 = \varphi(t_0)$. As a result, we obtain that a small piece γ of the curve Γ containing the point $A_0 = (\varphi(t_0), \psi(t_0), \chi(t_0))$ is described by two continuously differentiable function of x:

$$y = \psi[\varphi^{-1}(x)] = \psi_1(x),$$
$$z = \chi[\varphi^{-1}(x)] = \chi_1(x)$$

($c < x < d$, $c < x_0 < d$, $x_0 = \varphi(t_0)$). If, in fact, $\psi'(t_0) \neq 0$ or $\chi'(t_0) \neq 0$, then, reasoning in a similar

way, we obtain that some piece $\gamma \subset \Gamma$ is described by the equations

$$x = \varphi_1(y), \quad z = \chi_1(y)$$
$$(\lambda < y < \mu, \quad \lambda < y_0 < \mu, \quad y_0 = \psi(t_0))$$

or, respectively,

$$x = \varphi_1(z), \quad y = \psi_1(z)$$
$$(p < z < q, \quad p < z_0 < q, \quad z_0 = \chi(t_0)).$$

Equations (1) of the smooth curve Γ not only define (as the locus of points $(\varphi(t), \psi(t), \chi(t))$, $t \in [a, b]$), but also determine the *orientation* of Γ, i.e. the direction along which the parameter t increases. Figure 83 represents the smooth curve Γ corresponding to the variation of the parameter t on a closed interval $[a, b]$ $(a < b)$: $A = (\varphi(a), \psi(a), \chi(a))$ is the initial point of Γ, $B = (\varphi(b), \psi(b), \chi(b))$ is the terminal point of Γ, and the arrow indicates the orientation of Γ.

As the parameter t is continuously increasing from a to b, the point $(\varphi(t), \psi(t), \chi(t))$ continuously moves along Γ from the initial point $A = (\varphi(a), \psi(a), \chi(a))$ to the terminal point $B = (\varphi(b), \psi(b), \chi(b))$. The moving point can return to the initial position, i.e. it may happen that $t_1, t_2 \in [a, b]$, $t_1 < t_2$ and $\varphi(t_1) = \varphi(t_2)$, $\psi(t_1) = \psi(t_2)$, $\chi(t_1) = \chi(t_2)$, and then the curve Γ is said to be *self-crossing* (or *self-intersecting*). If the points A and B coincide, then Γ is a *closed curve*.

Let us introduce the function $t = \lambda(\tau)$, $c \leqslant \tau \leqslant d$, having a nonzero continuous derivative on $[c, d]$ and mapping $[c, d]$ onto $[a, b]$. Since $\lambda'(\tau)$ does not change sign on $[c, d]$, only two cases are possible:

(1) $\lambda'(\tau) > 0$, and then $\lambda(c) = a$, $\lambda(d) = b$,

(2) $\lambda'(\tau) < 0$, and then $\lambda(c) = b$, $\lambda(d) = a$.

Our smooth curve Γ can be defined by the equations

$$\left.\begin{aligned} x &= \varphi[\lambda(\tau)] = \varphi_1(\tau), \\ y &= \psi[\lambda(\tau)] = \psi_1(\tau), \\ z &= \chi[\lambda(\tau)] = \chi_1(\tau) \end{aligned}\right\} \qquad (1')$$

with the aid of the parameter τ $(c \leqslant \tau \leqslant d)$. One and the same smooth curve Γ can be represented parametrically by means of different parameters $t, \tau, \ldots$.

Note that conditions (2) in terms of τ are retained since, according to the formula for the derivative of a function of a function,

$$(\varphi_1'(\tau))^2 + (\psi_1'(\tau))^2 + (\chi_1'(\tau))^2$$
$$= [(\varphi'(t))^2 + (\psi'(t))^2 + (\chi'(t))^2] (\lambda'(\tau))^2 > 0. \quad (3)$$

But the introduction of the new parameter τ may cause a change in the orientation of Γ. If $\lambda'(\tau) > 0$ on $[c, d]$, then the function $t = \lambda(\tau)$ strictly increases and $\lambda(c) = a$, and $\lambda(d) = b$. In this case, with an increase in τ, t increases from $\lambda(c) = a$ to $\lambda(d) = b$, i.e. the orientation of Γ remains unchanged. Equations (1) and (1′) define one and the same smooth curve with the same orientation only with the aid of different parameters. And if $\lambda'(\tau) < 0$ on $[c, d]$, then $\lambda(c) = b$ and $\lambda(d) = a$ and with an increase in τ the parameter t decreases. In this case equations (1′) define the same curve Γ as do equations (1), but with an opposite orientation.

Whenever the orientation of a curve is concerned, by the letter Γ we understand not only the curve itself (the locus of points), but also its orientation. It should be borne in mind that equations (1) define both the curve itself and its orientation (the motion of a point in Γ towards the increased values of t). If we replace t by another parameter τ ($t = \lambda(\tau)$), then the same oriented curve Γ is obtained, provided $\lambda'(\tau) > 0$. And if $\lambda'(\tau) < 0$, we obtain the same curve but oppositely oriented. The latter should be denoted by a different symbol; Γ_- is just convenient for this purpose.

If the oriented curve Γ is defined by means of equations (1), then Γ_- can be, for example, represented by the equations

$$\left.\begin{aligned} x &= \varphi(-\tau), \\ y &= \psi(-\tau), \\ z &= \chi(-\tau), \end{aligned}\right\} \quad -b \leqslant \tau \leqslant -a.$$

Let us introduce the notion of arc length of a continuous curve Γ. Let a continuous curve Γ be described by means of equations (1). We partition the interval $[a, b]$ by the following values: $a = t_0 < t_1 < t_2 < \ldots < t_N = b$. To

each t_k there corresponds a point $A_k \in \Gamma$ ($A_0 = A$, $A_N = B$). We then connect the points A_k, in succession, by segments A_kA_{k+1} (Fig. 84) to obtain a polygonal line $\Gamma_N = A_0A_1 \ldots A_N$ inscribed in Γ. The length Γ_N is equal to the sum of the lengths $|A_kA_{k+1}|$:

$$|\Gamma_N| = \sum_{k=0}^{N-1} |A_kA_{k+1}|. \tag{4}$$

The limit of the length Γ_N, *as the maximum* $t_{j+1} - t_j$ *tends to zero*

$$\lim_{\max(t_{j+1}-t_j)\to 0} |\Gamma_N| = |\Gamma|, \tag{5}$$

provided it exists (is a finite number), *is called the arc length of* Γ. It will be denoted by $|\Gamma|$.

We can prove that for any continuous curve (1) there

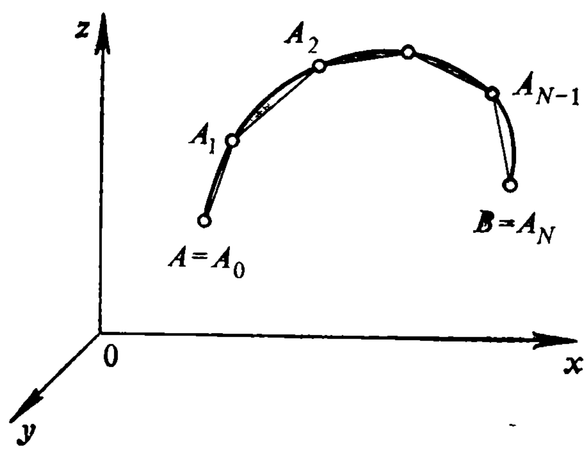

Fig. 84

exists limit (5) finite or infinite ($+\infty$). If this limit is finite, a curve is said to be *rectifiable.*

Theorem 1. *A smooth curve* Γ *defined by equalities* (1) *is rectifiable. Its arc length is equal to*

$$|\Gamma| = \int_a^b \sqrt{[\varphi'(t)]^2 + [\psi'(t)]^2 + [\chi'(t)]^2}\, dt. \tag{6}$$

It is of importance in this statement that the equations representing Γ are defined on a closed interval $[a, b]$.

If they were defined on an open interval (a, b), where φ, ψ, χ are continuously differentiable on (a, b) and their derivatives are not all zero, we would also say that equations (1) define a smooth curve, but the latter could also be not rectifiable. But any of its pieces corresponding to a closed interval $[c, d] \subset (a, b)$ is rectifiable.

The Proof of Theorem 1. Applying Lagrange's theorem to the functions φ, ψ, χ, we shall have ($\Delta t_k = t_{k+1} - t_k$, $\lambda_R = \max\limits_k \Delta t_k$)

$$\Delta\varphi = \varphi(t_{k+1}) - \varphi(t_k) = \varphi'(t_k')\,\Delta t_k,$$

$$\Delta\psi = \psi(t_{k+1}) - \psi(t_k) = \psi'(t_k'')\,\Delta t_k,$$

$$\Delta\chi = \chi(t_{k+1}) - \chi(t_k) = \chi'(t_k''')\,\Delta t_k,$$

and, consequently,

$$|\Gamma_N| = \sum_{k=0}^{N-1} \sqrt{\Delta^2\varphi + \Delta^2\psi + \Delta^2\chi}$$

$$= \sum_{k=0}^{N-1} \sqrt{[\varphi'(t_k')]^2 + [\psi'(t_k'')]^2 + [\chi'(t_k''')]^2}\,\Delta t_k$$

$$= \sum_{k=0}^{N-1} \sqrt{[\varphi'(t_k')]^2 + [\psi'(t_k')]^2 + [\chi'(t_k')]^2}\,\Delta t_k + r_N$$

$$\xrightarrow[\lambda_R \to 0]{} \int_a^b \sqrt{[\varphi'(t)]^2 + [\psi'(t)]^2 + [\chi'(t)]^2}\,dt \qquad (7)$$

(here $t_k', t_k'', t_k''' \in (t_k, t_{k+1})$ are, in general, different points), i.e. formula (6) is true.

Indeed, by virtue of the continuity of the integrand in (6),

$$\lim_{\lambda_R \to 0} \sum_{k=0}^{N-1} \sqrt{[\varphi'(t_k')]^2 + [\psi'(t_k')]^2 + [\chi'(t_k')]^2}\,\Delta t_k$$

$$= \int_a^b \sqrt{[\varphi'(t)]^2 + [\psi'(t)]^2 + [\chi'(t)]^2}\,dt.$$

Besides,

$$|r_N| = \left| \sum_{k=0}^{N-1} [\sqrt{[\varphi'(t'_k)]^2 + [\psi'(t''_k)]^2 + [\chi'(t'''_k)]^2} - \sqrt{[\varphi'(t'_k)]^2 + [\psi'(t'_k)]^2 + [\chi'(t'_k)]^2}]\, \Delta t_k \right|$$

$$\leqslant \sum_{k=0}^{N-1} \sqrt{[\psi'(t''_k) - \psi'(t'_k)]^2 + [\chi'(t'''_k) - \chi'(t'_k)]^2}\, \Delta t_k$$

$$\leqslant \frac{\varepsilon \sqrt{2}}{2(b-a)} \sum_{k=0}^{N-1} \Delta t_k < \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

Since the functions ψ' and χ' are continuous on $[a, b]$, they are also uniformly continuous on $[a, b]$. Therefore, if $\lambda_R < \delta$, then

$$|\psi'(t''_k) - \psi'(t'_k)| < \frac{\varepsilon}{2(b-a)}, \quad |\chi'(t'''_k) - \chi'(t'_k)| < \frac{\varepsilon}{2(b-a)}.$$

When estimating the quantity $|r_N|$, we took advantage of the inequality

$$|\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} - \sqrt{\eta_1^2 + \eta_2^2 + \eta_3^2}| \leqslant \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + (\xi_3 - \eta_3)^2},$$

showing that the difference of the lengths of two sides of a triangle does not exceed the length of its third side.

Let us apply formula (6) to compute the arc length of Γ when it is given by equations (1′) with the aid of the parameter τ. We have (see (6))

$$|\Gamma_1| = \int_c^d \sqrt{(\varphi_1'(\tau))^2 + (\psi_1'(\tau))^2 + (\chi_1'(\tau))^2}\, d\tau$$

$$= \int_c^d \sqrt{(\varphi'(\lambda(\tau)))^2 + (\psi'(\lambda(\tau)))^2 + (\chi'(\lambda(\tau)))^2}\, \lambda'(\tau)\, d\tau$$

$$= \int_a^b \sqrt{(\varphi'(t))^2 + (\psi'(t))^2 + (\chi'(t))^2}\, dt, \quad (\lambda'(\tau) > 0).$$

In the last equality of this chain we changed the variable $t = \lambda(\tau)$ in the integral.

Consequently, $|\Gamma_1| = |\Gamma|$.

We see that formula (6) for arc length is expressed in an invariant way in terms of the parameter of the arc.

Let us introduce the function

$$s = \mu(t) = \int_a^t \sqrt{(\varphi'(u))^2 + (\psi'(u))^2 + (\chi'(u))^2}\, du \quad (a \leqslant t \leqslant b) \tag{8}$$

of the upper limit of the integral. It expresses the length of the arc $\smile{AC}$, where C is a variable point of the arc $\smile{AB} = \Gamma$, corresponding to the value of the parameter t. The integrand in (8) is a continuous function of u, therefore the derivative of the arc length s with respect to t is equal to

$$\frac{ds}{dt} = \sqrt{(\varphi'(t))^2 + (\psi'(t))^2 + (\chi'(t))^2}. \tag{9}$$

Since $\varphi'(t)$, $\psi'(t)$, $\chi'(t)$ are continuous, ds/dt is, in its turn, a positive function of t (see Sec. 7.3, (2)). Then $s = \mu(t)$ strictly increases on $[a, b]$ and has an inverse which is a continuously differentiable function

$$t = \mu^{-1}(s), \quad 0 \leqslant s \leqslant |\Gamma|, \tag{10}$$

possessing the property

$$\frac{dt}{ds} = [(\varphi'(t))^2 + (\psi'(t))^2 + (\chi'(t))^2]^{-1/2} > 0.$$

But then the variable s can serve as the parameter of our smooth curve Γ and, hence, the equations of Γ can be written in the form

$$\left.\begin{aligned} x &= \varphi[\mu^{-1}(s)] = \varphi_*(s), \\ y &= \psi[\mu^{-1}(s)] = \psi_*(s), \\ z &= \chi[\mu^{-1}(s)] = \chi_*(s), \end{aligned}\right\} \quad 0 \leqslant s \leqslant |\Gamma|,$$

where the functions φ_*, ψ_*, χ_* are continuously differentiable on $[0, |\Gamma|]$.

In order to get appropriate results for the plane curve $\Gamma = \breve{AB}$ from the aforesaid, it is necessary to set $z = \chi(t) \equiv 0$. Then the smooth plane curve Γ is defined by two equations

$$\left.\begin{aligned} x &= \varphi(t), \\ y &= \psi(t), \end{aligned}\right\} \quad a \leqslant t \leqslant b,$$

where φ and ψ are continuously differentiable functions obeying the condition

$$(\varphi'(t))^2 + (\psi'(t))^2 > 0, \quad t \in [a, b].$$

The length of Γ is equal to

$$|\Gamma| = \int_a^b \sqrt{(\varphi'(t))^2 + (\psi'(t))^2}\, dt. \tag{6'}$$

The length of the arc $\breve{AC} \subset \Gamma$, where C is the point of Γ, corresponding to the value of the parameter $t \in [a, b]$

$$s = \int_a^t \sqrt{(\varphi'(u))^2 + (\psi'(u))^1}\, du, \tag{8'}$$

the differential of the arc being equal to

$$ds = \sqrt{(\varphi'(t))^2 + (\psi'(t))^2}\, dt. \tag{9'}$$

If Γ is defined with the aid of a continuously differentiable function

$$y = f(x), \quad a \leqslant x \leqslant b,$$

then we may consider that Γ is defined by the parameter x:

$$\left.\begin{aligned} x &= x, \\ y &= f(x), \end{aligned}\right\} \quad a \leqslant x \leqslant b.$$

Then by (6'),

$$|\Gamma| = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

And the differential of the arc Γ is expressed by the formula

$$ds = \sqrt{dx^2 + dy^2}.$$

Example 1. Find the arc length of the curve Γ: $y = \cosh x$, $0 \leqslant x \leqslant 2$.

We have

$$|\Gamma| = \int_0^2 \sqrt{1 + (\sinh x)^2}\, dx = \int_0^2 \cosh x \, dx = \sinh x \Big|_0^2 = \sinh 2 = \frac{e^2 - e^{-2}}{2}.$$

Example 2. Find the arc length of a circle Γ of radius R. The circle can be represented parametrically in the follow ing way:

$$\left.\begin{aligned} x &= R\cos t, \\ y &= R\sin t, \end{aligned}\right\} \quad 0 \leqslant t \leqslant 2\pi.$$

Then

$$|\Gamma| = \int_0^{2\pi} \sqrt{R^2 + \sin^2 t + R^2 \cos^2 t}\; dt = R \int_0^{2\pi} dt = 2\pi R.$$

Example 3. Find the arc length of the curve Γ: $y = \int_0^x \sqrt{2t + t^2}\, dt$, when x varies from 0 to 2.

Note that an implicit dependence of y on x can be found by computing the integral with the aid of an Euler substitution or regarding this integral as a linear fractional irrational. But in this case we do not need this implicit dependence. We have $y' = \sqrt{2x + x^2}$. Therefore

$$|\Gamma| = \int_0^2 \sqrt{1 + 2x + x^2}\, dx = \int_0^2 (x+1)\, dx = \frac{(x+1)^2}{2}\Big|_0^2 = 4.$$

Sec. 7.4. Curvature and Radius of Curvature of a Curve. Evolute and Evolvent

By the *curvature of a circle* of radius R is meant the number $1/R$. This number can be expressed as the ratio of the angle between the tangent lines drawn through

the end points of an arc of the circle to the length of that arc. The angle between the tangents to the circle at points A and B (see Fig. 85) is equal to the central angle α between the radii OA and OB. The length $|\overset{\smile}{AB}|$ of the arc $\overset{\smile}{AB}$ is equal to $R\alpha$. Therefore

$$\frac{\alpha}{|\overset{\smile}{AB}|}=\frac{\alpha}{R\alpha}=\frac{1}{R}.$$

This definition of the curvature of a circle provides an idea of a curvature which can be applied to an arbitrary smooth curve Γ.

Let us consider an arbitrary smooth curve Γ. As it was shown in Sec. 7.3, it is rectifiable and the length of its arbitrary arc $\overset{\smile}{AB}$ makes sense. The angle α $(0 \leqslant \alpha \leqslant \pi)$

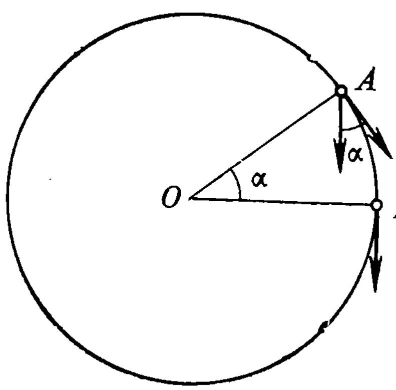

Fig. 85

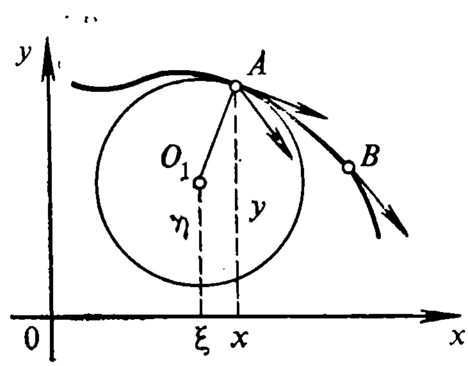

Fig. 86

between the positive directions of the tangent lines to the curve Γ at its points A and B is called the *angle of contingence of the arc* $\overset{\smile}{AB}$. The ratio of the angle of contingence (measured in radians) of the arc $\overset{\smile}{AB}$ to its length is called the *average curvature of the arc* $\overset{\smile}{AB}$ (Fig. 86). The *curvature* K *of the curve* Γ at its point A is the limit (finite or infinite) to which tends the ratio of the angle of contingence α of the arc $\overset{\smile}{AB}$ to its length $|\overset{\smile}{AB}| = |\Delta s|$, as the latter tends to zero:

$$K=\lim_{\Delta s\to 0}\frac{\alpha}{|\Delta s|}. \tag{1}$$

Thus, $0 \leqslant K \leqslant \infty$. By definition, the quantity $R = 1/K$ (here we agree that $0 = 1/\infty$ and $\infty = 1/0$) is called the *radius of curvature* of Γ at the point A.

The point O_1, lying at a distance $R = 1/K$ from the point A of Γ on the normal to Γ at the point A drawn in the direction of concavity of Γ, is called the *centre of*

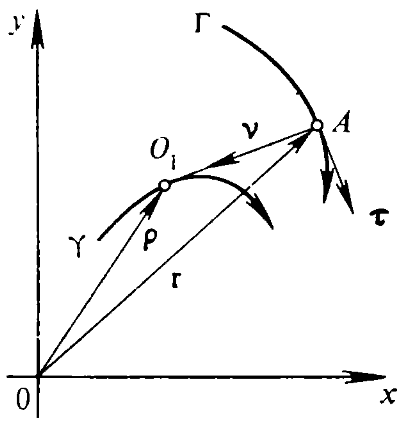

Fig. 87

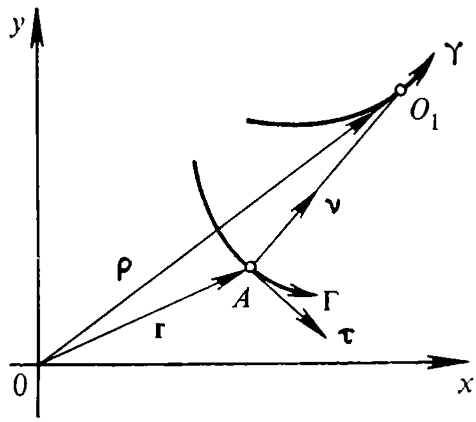

Fig. 88

curvature of Γ at the point A (Figs. 87 and 88). It is obvious that the centre of a circle coincides with the centre of its curvature.

The curve γ serving as the locus of the centres O_1 of curvature of a plane curve Γ is called the *evolute* of Γ. The curve Γ itself is termed an *evolvent* (*or involute*) of γ.

Let a curve Γ be defined by a function $y = f(x)$ $(c \leqslant x \leqslant d)$ having a continuous second derivative. We are going to find its curvature at the point $A = (x, f(x))$. Let φ_1 and φ_2 be angles formed by the tangent lines to Γ at the points A and $B = (x + \Delta x, f(x + \Delta x))$ with the positive direction of the x-axis (see Fig. 86)

$$\tan \varphi_1 = f'(x), \quad \tan \varphi_2 = f'(x + \Delta x),$$

$$\alpha = |\arctan f'(x) - \arctan f'(x + \Delta x)|. \tag{2}$$

Further,

$$\Delta s = |AB| = \int_x^{x+\Delta x} \sqrt{1 + (f'(u))^2}\, du. \tag{3}$$

Therefore, applying L'Hospital's rule (to Δx), we obtain from (1)

$$K=\lim_{\Delta x\to 0}\left|\frac{\arctan f'(x)-\arctan f'(x+\Delta x)}{\int\limits_x^{x+\Delta x}\sqrt{1+(f'(u))^2}\,du}\right|$$

$$=\lim_{\Delta x\to 0}\left|\frac{\dfrac{f''(x+\Delta x)}{1+(f'(x+\Delta x))^2}}{\sqrt{1+(f'(x+\Delta x))^2}}\right|=\frac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}.$$

Thus, we have obtained the formula for curvature:

$$K=\left|\frac{f''(x)}{(1+(f'(x))^2)^{3/2}}\right|. \tag{4}$$

If a smooth curve Γ is represented parametrically

$$\left.\begin{aligned} x&=\varphi(t),\\ y&=\psi(t),\end{aligned}\right\}\quad a\leqslant t\leqslant b,$$

where φ and ψ are twice continuously differentiable functions, then, using the rule for differentiation of functions represented parametrically, we obtain (see Sec. 4.11)

$$f'(x)=\frac{y'_t}{x'_t},\quad f''(x)=\frac{x'_ty''_t-y'_tx''_t}{(x'_t)^3},$$

$$R=\left|\frac{(x'^2_t+y'^2_t)^{3/2}}{y'_tx''_t-x'_ty''_t}\right|,\quad K=\frac{1}{R}. \tag{5}$$

Let us find the parametric equation of the evolute γ of a curve Γ defined by the equation $y=f(x)$ (Figs. 87-88). We have (see (4))

$$\frac{1}{R}=\frac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}=\frac{f''(x)\ \operatorname{sgn} f''(x)}{(1+(f'(x))^2)^{3/2}}. \tag{6}$$

Let the centre of curvature O_1 of the curve Γ at its point $A=(x, f(x))$ have the coordinates (ξ, η). It is determined by the vector

$$\boldsymbol{\rho}=\mathbf{r}+R\boldsymbol{\nu}, \tag{7}$$

where $\mathbf{r}$ is the radius vector of the point $A\in\Gamma$, and $\boldsymbol{\nu}$ is the unit vector of the normal in the direction of con-

cavity of Γ. The curve Γ has a vector equation

$$\mathbf{r} = (x,\ y).$$

Hence,

$$\dot{\mathbf{r}}_x = (1,\ y'_x), \quad \ddot{\mathbf{r}}_x = (0,\ y''_x).$$

Further (see Sec. 4.23, (3')),

$$\mathbf{v} = \pm\left(\frac{-y'_x}{\sqrt{1+y'^2_x}},\ \frac{1}{\sqrt{1+y'^2_x}}\right).$$

The sign should be chosen so that the vector $\mathbf{v}$ be directed towards the concavity of Γ, i.e. so that the scalar product $(\mathbf{v},\ \ddot{\mathbf{r}}_x)$ has the plus sign:

$$(\mathbf{v},\ \ddot{\mathbf{r}}_x) = \pm\frac{y''_x}{\sqrt{1+y'^2_x}} = y''_x\,(\operatorname{sgn} y''_x)\,(1+y'^2_x)^{-1/2}.$$

Thus,

$$\mathbf{v} = \operatorname{sgn} y''_x \cdot \left(\frac{-y'_x}{\sqrt{1+y'^2_x}},\ \frac{1}{\sqrt{1+y'^2_x}}\right). \tag{8}$$

Passing to the projections in equality (7) and taking into consideration (6) and (8), we obtain

$$\xi = x + \frac{(1+y'^2_x)^{3/2}}{y''_x \operatorname{sgn} y''_x} \cdot \frac{-y'_x \operatorname{sgn} y''_x}{(1+y'^2_x)^{1/2}} = x - \frac{y'_x\,(1+y'^2_x)}{y''_x},$$

$$\eta = y + \frac{(1+y'^2_x)^{3/2}}{y''_x \operatorname{sgn} y''_x}\,\frac{\operatorname{sgn} y''_x}{(1+y'^2_x)^{1/2}} = y + \frac{1+y'^2_x}{y''_x}. \tag{9}$$

It can be proved that the *normal to a curve* (*evolvent*) *at its point* $A = (x,\ f(x))$ *is a tangent line to the evolute* γ at the point $O_1 = (\xi,\ \eta)$. For this purpose it is sufficient to prove that the tangent lines to the curve Γ and to the evolute γ at the corresponding points are orthogonal

(perpendicular):

$$x'_x\xi'_x + y'_x\eta'_x = 1\cdot\left[1 - y''_x\,\frac{1+y'^2_x}{y''_x} - y'_x\left(\frac{1+y'^2_x}{y''_x}\right)'\right]$$
$$+\,y'_x\left[y'_x + \left(\frac{1+y'^2_x}{y''_x}\right)'\right] = 0.$$

Another important property of the evolute consists in the following. *The increment of the radius of curvature of an evolvent is equal up to a sign to the increment of the length of the corresponding arc of the evolute*:

$$R_2 - R_1 = \pm\,|\,\sigma_2 - \sigma_1\,|.$$

We leave the proof of this property to the reader.

Imagine that a flexible, inextensible string is stretched about the evolute. Now, let the string be unwound, under tension, from the evolute. At every moment in this process the string is tangent to the evolute and its free end point describes a curve which is nothing but an evolvent (Fig. 89). Since the string can be of arbitrary length, the given evolute generates an infinite number of evolvents. The length of the unwound portion of the string is, obviously, equal to the increment of the radius of curvature of the evolvent. If the curve Γ is represented parametrically: $x = x(t)$, $y = y(t)$, then the evolute is determined by the equations

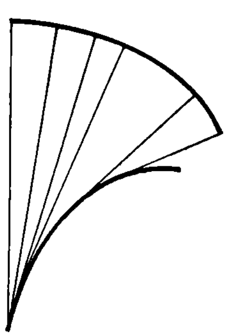

Fig. 89

$$\xi = x - y'_t\,\frac{x'^2_t + y'^2_t}{x'_t y''_t - y'_t x''_t}, \qquad \eta = y + x'_t\,\frac{x'^2_t + y'^2_t}{x'_t y''_t - y'_t x''_t} \tag{10}$$

(see Sec. 4.11).

Example 1. The evolute of the cycloid $x = t - \sin t$, $y = t - \cos t$ is the curve $\xi = t + \sin t$, $\eta = -1 + \cos t$. Setting $t = \tau + \pi$, we arrive at the equations

$$\xi - \pi = \tau - \sin\tau, \qquad \eta + 2 = 1 - \cos\tau,$$

We thus obtain the same curve (cycloid), but shifted relative to the original curve; thus, an evolute of the cycloid is a cycloid congruent to the given one (Fig. 90).

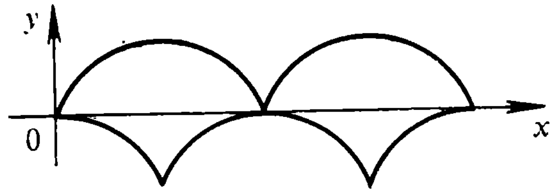

Fig. 90

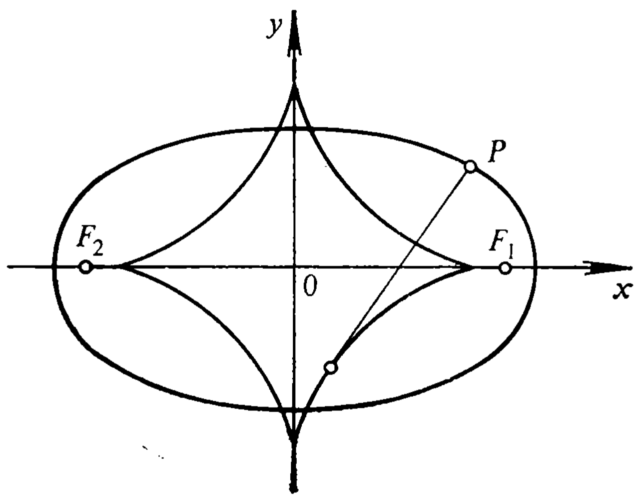

Fig. 91

Example 2. An evolute of the ellipse $x = a\cos t$, $y = b\sin t$ $(a \geqslant b > 0)$ is the astroid (Fig. 91)

$$\xi = \frac{a^2 - b^2}{a}\cos^3 t, \quad \eta = -\frac{a^2 - b^2}{b}\sin^3 t.$$

Sec. 7.5. Area of a Surface of Revolution

Let Γ be a plane curve described by a positive function $y = f(x)$ $(a \leqslant x \leqslant b)$ in a system of rectangular coordinates x, y and let f be continuously differentiable on $[a, b]$. We shall compute the area S of the surface of revolution generated by the rotation of Γ about the x-axis. To this end, we take a partition $a = x_0 < x_1 < \ldots$ $\ldots < x_n = b$ of the interval $[a, b]$, inscribe in the

curve Γ the polygonal line Γ_n with vertices at the points $(x_k, f(x_k))$, and compute the area of the surface of revolution generated by the rotation of the polygonal line about the x-axis (which is the sum of areas of lateral surfaces of truncated cones):

$$S_n = \pi \sum_{k=0}^{n-1} [f(x_k) + f(x_{k+1})] \sqrt{\Delta x_k^2 + \Delta y_k^2},$$

$$\Delta y_k = f(x_{k+1}) - f(x_k).$$

Applying Lagrange's theorem to the difference Δy_k, we get

$$S_n = \pi \sum_{k=0}^{n-1} [f(x_k) + f(x_{k+1})] \sqrt{1 + (f'(\xi_k))^2}\, \Delta x_k$$

$$= 2\pi \sum_{k=0}^{n-1} f(x_k) \sqrt{1 + (f'(x_k))^2}\, \Delta x_k + R_n$$

$$\to 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2}\, dx$$

for $\lambda_R = \max_k \Delta x_k \to 0$; according to Lagrange's theorem, the point $\xi_k \in (x_k, x_{k+1})$. Indeed, since f and f' are continuous on $[a, b]$, the function $f(x) \sqrt{1 + f'(x)^2}$ is integrable, therefore

$$\lim_{\lambda_k \to 0} 2\pi \sum_{k=0}^{n-1} f(x_k) \sqrt{1 + (f'(x_k))^2}\, \Delta x_k$$

$$= 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2}\, dx.$$

Further,

$$|R_n| = \left| \pi \sum_{k=0}^{n-1} \{[f(x_k) + f(x_{k+1})] \sqrt{1 + (f'(\xi_k))^2}\right.$$

$$\left. - 2f(x_k) \sqrt{1 + (f'(x_k))^2}\} \Delta x_k \right|$$

$$= \left| \pi \sum_{k=0}^{n-1} \{(f(x_k) + f(x_{k+1})) \left(\sqrt{1 + (f'(\xi_k))^2} - \sqrt{1 + (f'(x_k))^2}\right)\right.$$

$$+ (f(x_{k+1}) - f(x_k)) \sqrt{1 + (f'(x_k))^2}\} \Delta x_k |$$

$$\leqslant \pi \sum_{k=0}^{n-1} \{(|f(x_k)| + |f(x_{k+1})|) \, |f'(\xi_k) - f'(x_k)|$$

$$+ \sqrt{1 + (f'(x_k))^2} \, |f(x_{k+1}) - f(x_k)|\} \Delta x_k.$$

Since f and f' are continuous on $[a, b]$, they are bounded and uniformly continuous on $[a, b]$. Therefore, $|f| \leqslant M$, $\sqrt{1 + (f'(x))^2} \leqslant M$, and $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|f(x_{k+1}) - f(x_k)| < \frac{\varepsilon}{3\pi M(b-a)},$$

$$|f'(x_k) - f'(\xi_k)| \leqslant \frac{\varepsilon}{3\pi M(b-a)},$$

for $\lambda_R < \delta$. Hence,

$$|R_n| \leqslant \pi \sum_{k=0}^{n-1} \left\{ 2M \frac{\varepsilon}{3\pi M(b-a)} + M \frac{\varepsilon}{3\pi M(b-a)} \right\} \Delta x_k$$

$$= \frac{\varepsilon}{b-a} \sum_{k=0}^{n-1} \Delta x_k = \frac{\varepsilon}{b-a}(b-a) = \varepsilon,$$

i.e. $R_n \to 0$ as $\lambda_R \to 0$. Thus, the area of the surface of the solid of revolution is equal to

$$S = 2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2} \, dx. \tag{1}$$

Example. Find the area S of the surface of revolution of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)$$

about the x-axis (the area of the surface of an ellipsoid of revolution).

Solution. The equation of the upper half of the ellipse

$$y=\frac{b}{a}\sqrt{a^2-x^2}\ (|x|\leqslant a),\quad y'=-\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}},$$

$$S=2\pi\int_{-a}^{a}\frac{b}{a}\sqrt{a^2-x^2}\sqrt{1+\left(\frac{b}{a}\right)^2\cdot\frac{x^2}{a^2-x^2}}\,dx$$

$$=\frac{4\pi b}{a^2}\int_0^a\sqrt{a^4-(a^2-b^2)x^2}\,dx=(u=x\sqrt{a^2-b^2})$$

$$=\frac{4\pi b}{a^2\sqrt{a^2-b^2}}\int_0^{a\sqrt{a^2-b^2}}\sqrt{a^4-u^2}\,du$$

$$=\frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left\{\frac{u}{2}\sqrt{a^4-u^2}+\frac{a^4}{2}\arcsin\frac{u}{a^2}\right\}_0^{a\sqrt{a^2-b^2}}$$

$$=2\pi b^2+\frac{2\pi ba^2}{\sqrt{a^2-b^2}}\arcsin\frac{\sqrt{a^2-b^2}}{a}.$$

As $b\to a$, we obtain in the limit that $S=4\pi a^2$, i.e. the area of the surface of a sphere of radius a.

Sec. 7.6. Lagrange's Interpolation Formula

Let us consider the following problem: it is required to find an algebraic polynomial $L_n(x)$ of degree n which would coincide with a function $f(x)$ at given points x_0, $x_1, \ldots, x_n$. Thus, the following conditions must be fulfilled:

$$f(x_k)=L_n(x_k)\quad (k=0,1,\ldots,n).$$

The polynomial $L_n(x)$ is unique. If we assume that there exists one more polynomial $\overline{L_n}(x)$ with the same properties, then the difference $L_n(x)-\overline{L_n}(x)$ will vanish at $(n+1)$ points $x_0, \ldots, x_n$ and be an algebraic polynomial of degree n. Hence, the difference is identically equal to zero and $L_n(x)\equiv\overline{L_n}(x)$.

It follows from the uniqueness that if the original function $f(x)$ is itself an algebraic polynomial of degree n, then it coincides with $L_n(x)$ for all x's ($f(x)\equiv L_n(x)$).

We shall first find the algebraic polynomial of degree n $Q_{n,k}(x)$ which is equal to zero at points $x_i \neq x_k$ and to unity at point x_k. It is obvious that

$$Q_{n,k}(x) = A(x - x_0) \ldots (x - x_{k-1})(x - x_{k+1}) \ldots \ldots (x - x_n),$$

where the constant A is found from the condition

$$1 = Q_{n,k}(x_k) = A \prod_{\substack{i=0 \\ i \neq k}}^{n} (x_k - x_i), \quad \text{i.e.} \quad A = \prod_{\substack{i=0 \\ i \neq k}}^{n} (x_k - x_i)^{-1}.$$

Hence, the required polynomial has the form

$$Q_{n,k}(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}.$$

Introducing the Kronecker delta*

$$\delta_{ki} = \begin{cases} 0, & k \neq i, \\ 1, & k = i, \end{cases}$$

we get

$$Q_{n,k}(x_i) = \delta_{ki}.$$

Our problem is solved by the polynomial

$$L_n(x) = \sum_{k=0}^{n} Q_{n,k}(x) f(x_k), \tag{1}$$

since

$$L_n(x_i) = \sum_{k=0}^{n} Q_{n,k}(x_i) f(x_k) = \sum_{k=0}^{n} \delta_{ki} f(x_k) = f(x_i)$$

$$(i = 0, 1, \ldots, n).$$

Polynomial (1) is called the *Lagrange interpolation polynomial.*

The same as in deriving the formula for the remainder in Taylor's formula, we can show that if $f(x)$ has a deriv-

* The symbol δ_{ki} is called the Kronecker delta after Leopold Kronecker (1823-1891), a German mathematician.—*Tr.*

ative of order $(n+1)$, then

$$f(x) - L_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,\omega_{n+1}(x), \tag{2}$$

where

$$\omega_{n+1}(x) = \prod_{i=0}^{n} (x - x_i)$$

and ξ is a certain point belonging to the smallest interval containing the points $x_0, x_1, \ldots, x_n, x$. Indeed, let us set

$$f(x) - L_n(x) = K\omega_{n+1}(x), \tag{3}$$

where K is a quantity depending on x. Let us introduce the following notation:

$$\varphi(z) = f(z) - L_n(z) - K\omega_{n+1}(z),$$

where K has the same value as in (3); this is a quantity independent of z. It is clear that $\varphi(x) = 0$, $\varphi(x_i) = 0$ $(i = 0, 1, \ldots, n)$. Let, for instance, $x_0 < x < x_1 < \ldots < x_n$, then, applying Rolle's theorem to the function φ on the intervals $[x_0, x]$, $[x, x_1]$ $\ldots$, $[x_{n-1}, x_n]$, we obtain that the derivative $\varphi'(x)$ vanishes inside each of them. Then, applying Rolle's theorem, in succession, to the functions $\varphi', \ldots, \varphi^{(n)}$, we conclude that there exists a point ξ belonging to the smallest interval containing the points $x, x_0, x_1, \ldots, x_n$ at which $\varphi^{(n+1)}(\xi) = 0$, but

$$\varphi^{(n+1)}(z) = f^{(n+1)}(z) - K(n+1)!.$$

Setting $z = \xi$, we get $K = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}$. Therefore,

$$f(x) - L_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,\omega_{n+1}(x),$$

and equality (2) has been proved.

The Lagrange interpolation polynomial is used for approximate computation of the derivatives of the function $f(x)$ when its values are known only at the points x_0, $x_1, \ldots, x_n$; namely, we set

$$f^{(k)}(x) \approx L_n^{(k)}(x).$$

For example, if $f(x)$ is known at the points x_0, x_1, then, constructing the Lagrange polynomial $L_1(x)$ for these points, we find that

$$f'(x) \approx \frac{f(x_1)-f(x_0)}{x_1-x_0}.$$

In the subsequent sections we shall consider the application of Lagrange's polynomial to approximate computation of the definite integral.

Sec. 7.7. Rectangle and Trapezoid Quadrature Formulas

Let it be required to compute the definite integral of a continuous function f on a closed interval $[a, b]$. If its antiderivative is known then we can naturally apply the Newton-Leibniz formula. But, by far not always an antiderivative of a continuous function can be found analytically, and therefore there arises the problem of an approximate computation of the integral.

The simplest method for approximate computation of a definite integral is directly implied by its definition. Let us divide the interval of integration $[a, b]$ into equal parts with the aid of the points

$$x_k = a + k\frac{b-a}{N} \quad (k = 0, 1, \ldots, N) \tag{1}$$

and set

$$\int_a^b f(x)\,dx \approx \frac{b-a}{N}\sum_{k=0}^{N-1} f\left(\frac{x_k+x_{k+1}}{2}\right), \tag{2}$$

where the sign $\approx$ indicates an approximate equality.

Relation (2) is referred to as the *rectangle formula* (or the rectangle rule; formulas of this kind are termed *quadrature formulas*). Figure 92 shows that in the case of a positive function $f(x)$ the sought-for area of the figure bounded by the curve $y = f(x)$, by the x-axis, and by the straight lines $x = a$ and $x = b$ is approximately equal to the sum of the areas of the rectangles seen in the figure.

We know that in the case of a continuous function f on $[a, b]$ the limit, as $N \to \infty$, of the right-hand member of approximate equality (2) is exactly equal to its left-hand member; therefore it is natural to expect that the error of the approximate formula (2), that is, the absolute value of the difference between its right-hand

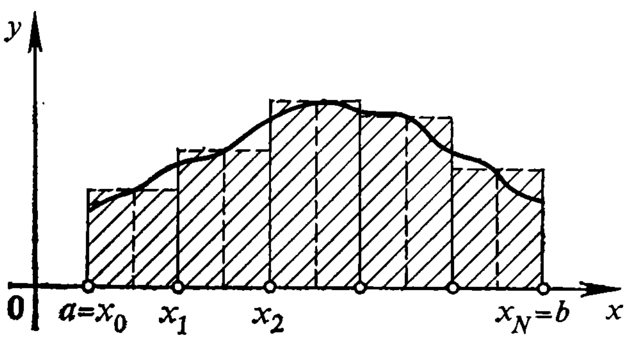

Fig. 92

and left-hand members, is small for large N's. In this connection there arises the question as to how the error of such an approximation can be estimated. As will be shown, such an estimation can readily be found if we require additionally that the function f (which is supposed to be continuous) satisfy some smoothness conditions.

It is important to stress that if f is a linear function of the form $f(x) = Ax + B$, *formula* (2) *yields an exact equality*: in this case the right-hand side of (2) exactly coincides with its left-hand side. Since a linear function is a polynomial of the first degree, we can say that the *rectangle formula provides an exact result for all polynomials of degree not higher than* 1.

Another natural method for computing approximately a definite integral leads to the so-called *trapezoid* (*quadrature*) *formula* (*trapezoid rule*). According to this method, the interval of integration is partitioned into equal parts with the aid of a system of points of type (1) and the approximate expression for the sought-for integral

is written as

$$\int_a^b f(x)\,dx \approx \frac{b-a}{N}\left(\frac{f(x_0)+f(x_1)}{2}+\frac{f(x_1)+f(x_2)}{2}\right.$$
$$\left. + \dots + \frac{f(x_{N-1})+f(x_N)}{2}\right.$$
$$= \frac{b-a}{2N}[f(x_0)+2f(x_1)+2f(x_2)+\dots$$
$$+2f(x_{N-1})+f(x_N)]. \quad (3)$$

As is seen from Fig. 93, the trapezoid formula expresses approximately the sought-for area as the sum of the areas of the trapezoids shown in the figure. It is important to

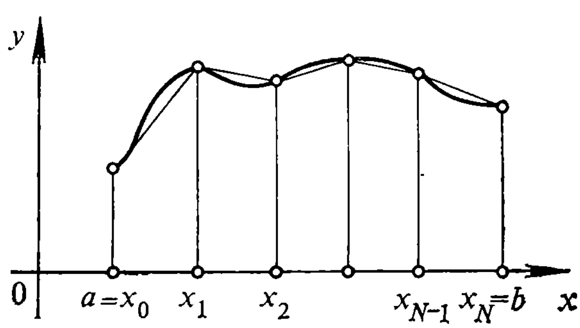

Fig. 93

note that the *trapezoid formula provides an exact result for any linear function* $Ax + B$ (where A and B are constants), that is, for any polynomial of degree not higher than 1; if an expression $Ax + B$ is substituted for $f(x)$ into (3), then this relation becomes exact. In this sense the trapezoid formula has no advantages over the rectangle formula, since they are both exact for linear functions.

Let us denote the difference between the left-hand and right-hand members of a quadrature formula by $R_N(f)$ and call it the *remainder of the quadrature formula.*

If a function f has a piecewise continuous derivative f' satisfying the inequality $|f'(x)| \leqslant M_1$, then the remainder of rectangle formula (2) obeys the inequality

$$|R_N(f)| \leqslant \frac{M_1(b-a)^2}{4N}, \quad (2')$$

and the remainder of trapezoid formula (3) obeys the inequality

$$|R_N(f)| \leqslant \frac{M_1 (b-a)^2}{4N}. \tag{3'}$$

It should be noted that here the constants are computed accurately, therefore they are not allowed to be reduced. Estimate (2′) is derived below, the remaining estimates being given without proofs.

We see that in both cases for the class of functions having a bounded derivative $|f'(x)| \leqslant M_1$, the corresponding remainders have the order $O(N^{-1})$ (see Sec. 3.10, (14)).

And for the class of functions having a bounded second derivative $|f''(x)| \leqslant M_2$ on $[a, b]$, the following estimate takes place

$$|R_N(f)| \leqslant \frac{M_2 (b-a)^3}{12N^2},$$

which is true for the rectangle and trapezoid formulas. Now the order of approximation by means of both quadrature formulas in question is $O(N^{-2})$.

It turns out that for the class of functions having a bounded derivative of order $l > 2$ the order of approximation by means of the rectangle and trapezoid formulas does not improve, it remains equal to $O(N^{-2})$.

This is explained by the fact that both quadrature formulas are exact for polynomials of the first degree, but they are inexact for polynomials of degree higher than 1.

If a function f has a bounded third derivative, then it is possible to derive a quadrature formula giving an approximation error of order $O(N^{-3})$. This formula must be exact for polynomials of the second degree. But if it is inexact for polynomials of the third degree, then for the functions having a bounded derivative of the fourth order the approximation error remains of the order equal to $O(N^{-3})$. The phenomenon described here will be illustrated by a particular example of Simpson's* quadrature formula in the next section.

Let us now prove estimate (2′). To this end, we introduce the following notation: $h = \frac{b-a}{N}$, $\xi_k = \frac{x_k + x_{k+1}}{2}$, x_k are points of system (1). Then

$$|R_N(f)| = \left| \int_a^b f(x)\,dx - h \sum_0^{N-1} f(\xi_k) \right|$$

$$= \left| \sum_0^{N-1} \int_{\xi_k - \frac{h}{2}}^{\xi_k + \frac{h}{2}} f(x)\,dx - h \sum_0^{N-1} f(\xi_k) \right|$$

---

* Simpson, Thomas (1710-1761). English mathematician.

$$\leqslant \sum_{0}^{N-1} \left| \int_{\xi_k - \frac{h}{2}}^{\xi_k + \frac{h}{2}} f(x)\,dx - hf(\xi_k) \right| = \sum_{0}^{N-1} \left| \int_{\xi_k - \frac{h}{2}}^{\xi_k + \frac{h}{2}} [f(x) - f(\xi_k)]\,dx \right|,$$

since

$$\int_{\xi_k - \frac{h}{2}}^{\xi_k + \frac{h}{2}} f(\xi_k)\,dx = hf(\xi_k).$$

Applying Lagrange's theorem to what stands under the integral sign and taking into account that $|f'(x)| \leqslant M_1$, we obtain

$$|R_N(f)| \leqslant \sum_{0}^{N-1} \left| \int_{\xi_k - \frac{h}{2}}^{\xi_k + \frac{h}{2}} f'(\theta_k)(x - \xi_k)\,dx \right|$$

$$\leqslant \sum_{0}^{N-1} \int_{\xi_k - \frac{h}{2}}^{\xi_k + \frac{h}{2}} |f'(\theta_k)|\,|x - \xi_k|\,dx \leqslant M_1 \sum_{0}^{N-1} \int_{\xi_k - \frac{h}{2}}^{\xi_k + \frac{h}{2}} |x - \xi_k|\,dx,$$

where θ_k is a point lying between x and ξ_k. Changing the variable $(x - \xi_k = t)$, we obtain

$$R_N(f) | \leqslant M_1 \sum_{0}^{N-1} \int_{-h/2}^{h/2} |t|\,dt = 2M_1 \cdot N \int_{0}^{h/2} t\,dt$$

$$= 2M_1 N \left. \frac{t^2}{2} \right|_0^{h/2} = \frac{M_1 N h^2}{4} = \frac{M_1 (b-a)^2}{4N}.$$

Sec. 7.8. Simpson's Formula

Let it be required to approximately compute the integral of a continuous function $f(x)$:

$$\int_a^b f(x)\,dx. \tag{1}$$

We shall look for an approximate value of the integral in the form of a sum

$$\int_a^b f(x)\,dx \approx \sum_{k=0}^{n} p_k f(x_k), \qquad (2)$$

where $p_0, p_1, \ldots, p_n$ and $x_0, x_1, \ldots, x_n \in [a, b]$ are given numbers.

Formula (2) is called the quadrature formula *with nodes* x_k and *weights* p_k.

When constructing concrete approximation formulas, we require that formula (2) be exact for algebraic polynomials of degree n. This requirement will be fulfilled if we take the definite integral of Lagrange's interpolation polynomial of the nth degree of the function f as an approximate value of integral (1):

$$\int_a^b f(x)\,dx \approx \int_a^b L_n(x)\,dx$$

$$= \int_a^b \sum_{k=0}^{n} Q_{n,k}(x) f(x_k)\,dx = \sum_{k=0}^{n} p_k f(x_k), \qquad (3)$$

$$p_k = \int_a^b Q_{n,k}(x)\,dx \quad (k = 0, 1, \ldots, n),$$

$$Q_{n,k}(x) = \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)},$$

since, if $f(x)$ is a polynomial of degree n, then $f(x) \equiv$ $\equiv L_n(x)$.

Let us obtain formula (3) for the case $n = 2$ and the nodes $x_0 = a$, $x_1 = \frac{a+b}{2}$, $x_2 = b$. In this case

$$Q_{2,0}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-b)(2x-a-b)}{(b-a)^2}$$

$$= \frac{2(x-b)^2}{(b-a)^2} + \frac{x-b}{b-a},$$

$$Q_{2,1}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{-4(x-a)(x-b)}{(b-a)^2}$$

$$= -4\frac{(x-b)^2}{(b-a)^2} - 4\frac{x-b}{b-a},$$

$$Q_{2,2}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-a)(2x-a-b)}{(b-a)^2}$$

$$= \frac{2(x-a)^2}{(b-a)^2} - \frac{x-a}{b-a}.$$

Therefore,

$$p_0 = \int_a^b Q_{2,0}(x)\,dx = \left[\frac{2(x-b)^3}{3(b-a)^2} + \frac{(x-b)^2}{2(b-a)}\right]_a^b = \frac{b-a}{6}.$$

Reasoning in a similar way, we obtain

$$p_1 = \int_a^b Q_{2,1}(x)\,dx = \frac{2(b-a)}{3}, \quad p_2 = \int_a^b Q_{2,2}(x)\,dx = \frac{b-a}{6}.$$

By virtue of this, formula (3), for $n = 2$, has the form

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]. \qquad (4)$$

This is a simplest Simpson's quadrature formula corresponding to the closed interval $[a, b]$.

From the geometrical point of view, formula (4) means

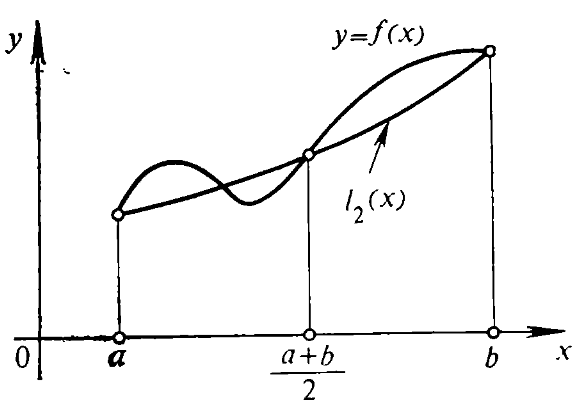

Fig. 94

that we have replaced the area of the curvilinear trapezoid defined by the function $f(x)$ on $[a, b]$ by the area under the graph of a parabola (see Fig. 94):

$$y = L_2(x) = f(a)\,Q_{2,0}(x) + f\left(\frac{a+b}{2}\right)Q_{2,1}(x) + f(b)\,Q_{2,2}(x).$$

We want to note once again that, by construction, formula (4) is exact for polynomials of the second degree. But it turns out that it is exact for polynomials of the third degree as well. Indeed, $\int\limits_a^b x^3\,dx = \frac{b^4-a^4}{4}$ and the right-hand member of formula (4) for the function $f(x) = x^3$ is also equal to this number:

$$\frac{b-a}{6}\left[a^3+4\left(\frac{a+b}{2}\right)^3+b^3\right]$$

$$=\frac{b-a}{6}\left[(a+b)(a^2-ab+b^2)+\frac{(a+b)^3}{2}\right]$$

$$=\frac{b^2-a^2}{6}\left[\frac{3(a^2+b^2)}{2}\right]=\frac{b^4-a^4}{4}.$$

Thus, formula (4) is *exact for polynomials of degree not higher than the third.*

If we separate the interval $[a, b]$ into $2N$ equal parts with the aid of the points of division

$$x_k = a+\frac{b-a}{2N}k \quad (k=0, 1, \ldots, 2N)$$

and apply formula (4) to each of the subintervals $[x_0, x_2]$, $[x_2, x_4]$, . . ., then the addition of the results obtained leads to *Simpson's complicated quadrature formula*

$$\int\limits_a^b f(x)\,dx \approx \frac{b-a}{6N}\{f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)$$

$$+\ldots+2f(x_{2N-2})+4f(x_{2N-1})+f(x_{2N})\}. \quad (5)$$

From the point of view of practical applications the amounts of calculations involved in Simpson's formula and in the rectangle formula are essentially the same. But if the integrand function f is sufficiently smooth then, for large N's, the error of the approximation computed by Simpson's formula is considerably smaller than the corresponding error introduced by the rectangle formula.

If a function $f(x)$ has on $[a, b]$ a continuous second derivative satisfying the inequality

$$|f''(x)| \leqslant M_2,$$

and has no third derivative or we cannot estimate it for some reason, then the integral $\int_a^b f(x)\,dx$ is recommended to be computed by the trapezoid formula or, which is still better, by Simpson's formula.

It can be proved that the error of the approximation obtained by the trapezoid formula (Sec. 7.7, (3)) will be:

$$\frac{1}{12}\cdot\frac{(b-a)^3}{N^2}M_2 \tag{6}$$

and by Simpson's formula (5):

$$\frac{1}{81}\,\frac{(b-a)^3}{N^2}M_2.$$

If a function $f(x)$ has on $[a, b]$ a continuous fourth derivative satisfying the inequality

$$|f^{(4)}(x)| \leqslant M_4,$$

then it is recommended to apply Simpson's formula. In this case, the approximation error will be equal to

$$\frac{1}{2880}\,\frac{(b-a)^5}{N^4}M_4. \tag{7}$$

If the trapezoid formula was applied in this case, then the approximation error would have, as before, the order N^{-2}, i.e. would be worse than (7).

Example 1. Compute the integral $I=\int_0^1 \sqrt{1+x^4}\,dx$.

The given integral (of binomial differential) is not expressible in elementary functions.

Let us evaluate this integral approximately, by dividing the closed interval [0, 1] into ten equal parts and making use of various quadrature formulas. Denoting the points of division by $x_0 = 0$, $x_1 = 0.1, \ldots, x_9 = 0.9$, $x_{10} = 1$, we compute the approximate values of the function $f(x) = \sqrt{1+x^4}$ at these points:

$$f(0) = 1, \quad f(x_1) = 1.00005, \quad f(x_2) = 1.00080,$$
$$f(x_3) = 1.00404, \quad f(x_4) = 1.01272, \quad f(x_5) = 1.03078,$$
$$f(x_6) = 1.06283, \quad f(x_7) = 1.11360, \quad f(x_8) = 1.18727,$$
$$f(x_9) = 1.28690, \quad f(x_{10}) = 1.41421.$$

According to the trapezoid quadrature formula, we have

$$I \approx \frac{1}{20}[f(x_0)+2f(x_1)+\ldots+2f(x_9)+f(x_{10})]$$
$$=\frac{21.81219}{20}=1.09061.$$

The function $f(x) = \sqrt{1+x^4}$ has arbitrary many continuous derivatives on the interval [0, 1]. As was noted above, the existence of derivatives of order higher than the second does not influence the exactness of the trapezoid formula. Therefore we shall determine the error of the trapezoid formula proceeding from the fact of existence of a continuous second derivative

$$f''(x) = 2x^2(3+x^4)/(1+x^4)^{3/2}.$$

Since $M_2 = \max\limits_x f''(x) = 2\sqrt{2}$, the remainder of the trapezoid formula

$$R_{10}(f) \leqslant \frac{M_2}{12N^2} = \frac{2\sqrt{2}}{12\times 10^2} \approx 0.002357.$$

Hence

$$I = 1.0906 \pm 0.0024.$$

By Simpson's formula $(2N = 10)$

$$I \approx \frac{1}{30}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5)$$

$$+ 2f(x_6) + 4f(x_7) + 2f(x_8) + 4f(x_9) + f(x_{10})] = \frac{32.68473}{30} = 1.08949.$$

The remainder of Simpson's formula can be determined by taking into account that $f(x)$ has a continuous derivative of the fourth order (the existence of derivatives of order higher than the fourth does not affect the exactness of Simpson's formula)

$$f^{(4)}(x) = 12(1 - 14x^4 + 5x^8)/(1+x^4)^{7/2}.$$

Since $M_4 = \max\limits_x |f^{(4)}(x)| \leqslant 15\sqrt{2}$, we have

$$R_5(f) \leqslant \frac{M_4}{2880N^4} \leqslant \frac{15\sqrt{2}}{2880\times 5^4} \leqslant 0.000012 < 0.00002.$$

Hence,

$$I = 1.08949 \pm 0.00002,$$

i.e. Simpson's formula is considerably more exact than the trapezoid formula for sufficiently smooth functions and a large N.

Remark. All the computations were carried out with the aid of a pocket microcomputer, type "Electronika BZ-18M".

CHAPTER 8

DIFFERENTIAL CALCULUS. FUNCTIONS OF SEVERAL VARIABLES

Sec. 8.1. Preliminaries

The notion of a function of several variables was introduced at the end of Sec. 3.1. In order to study the functions $z = f(\mathbf{x}) = f(x_1, \ldots, x_n)$ of n variables, i.e. the functions of the points $\mathbf{x} = (x_1, \ldots, x_n)$ of an n-dimensional space R_n ($\mathbf{x} \in R_n$), the reader must know the fundamentals of an n-dimensional space*.

The distance between two points $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{x}' = (x'_1, \ldots, x'_n)$ of the n-dimensional space $R_n = R$ is determined by the formula

$$|\mathbf{x}-\mathbf{x}'| = \sqrt{\sum_{j=1}^{n} (x_j - x'_j)^2}.$$

Note that if $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are three points in the space R, then the following inequalities are fulfilled for them:

$$|\mathbf{x}-\mathbf{y}| \leqslant |\mathbf{x}-\mathbf{z}| + |\mathbf{y}-\mathbf{z}|, \qquad (1)$$

$$||\mathbf{x}| - |\mathbf{y}|| \leqslant |\mathbf{x}-\mathbf{y}|. \qquad (2)$$

They are called the *triangle inequalities*. In a three-dimensional Euclidean space they have a geometrical interpretation: the points $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ determine a triangle with the vertices at these points and sides having the lengths $|\mathbf{x}-\mathbf{y}|$, $|\mathbf{x}-\mathbf{z}|$, $|\mathbf{y}-\mathbf{z}|$. It is known from geometry that the length of a side of a triangle does not exceed the sum of the lengths of its other two sides, and the difference of the lengths of two sides of a triangle does not exceed the length of its third side. This is just written in the form of inequalities (1) and (2).

---

* See our book *Fundamentals of Linear Algebra and Analytical Geometry*, Sec. 6.

In terms of the coordinates of the points $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ inequality (1) is written in the following way:

$$\left(\sum_{j=1}^{n}(x_j-y_j)^2\right)^{1/2} \leqslant \left(\sum_{j=1}^{n}(x_j-z_j)^2\right)^{1/2} + \left(\sum_{j=1}^{n}(z_j-y_j)^2\right)^{1/2}. \tag{1'}$$

This is *Minkowski's inequality.*

Inequality (2) directly follows from inequality (1). Indeed,

$$|\mathbf{x}| = |\mathbf{x}-\mathbf{y}+\mathbf{y}| \leqslant |\mathbf{x}-\mathbf{y}| + |\mathbf{y}|,$$
$$|\mathbf{y}| = |\mathbf{y}-\mathbf{x}+\mathbf{x}| \leqslant |\mathbf{x}-\mathbf{y}| + |\mathbf{x}|,$$

therefore

$$-|\mathbf{x}-\mathbf{y}| \leqslant |\mathbf{x}| - |\mathbf{y}| \leqslant |\mathbf{x}-\mathbf{y}|,$$

i.e. we have (2).

By definition, a sequence of points

$$\mathbf{x}^1 = (x_1^1, \ldots, x_n^1),$$
$$\mathbf{x}^2 = (x_1^2, \ldots, x_n^2),$$
$$\mathbf{x}^3 = (x_1^3, \ldots, x_n^3),$$
$$\cdots\cdots\cdots\cdots$$

of the space R *tends* (*converges*) *to the point* $\mathbf{x}^0 \in R$, if

$$|\mathbf{x}^k - \mathbf{x}^0| \to 0 \quad (k \to \infty), \tag{3}$$

i.e. if

$$\sqrt{\sum_{j=1}^{n}(x_j^k - x_j^0)^2} \to 0. \tag{3'}$$

In this case we write

$$\lim \mathbf{x}^k = \lim_{k\to\infty} \mathbf{x}^k = \mathbf{x}^0 \quad \text{or} \quad \mathbf{x}^k \to \mathbf{x}^0$$

and say that the point $\mathbf{x}^0$ is the limit of a sequence of points $\mathbf{x}^k$. From (3′) it follows

$$x_j^k \to x_j^0, \quad k \to \infty \quad (j = 1, \ldots, n) \tag{3''}$$

and conversely. Thus, to say that a sequence of points $\mathbf{x}^k$ converges to the point $\mathbf{x}^0$ is all the same as to say that

each coordinate x_j^k of a variable point $\mathbf{x}^k$ converges to the corresponding coordinate x_j^0 of the point $\mathbf{x}^0$.

The following properties are valid:

$$\lim (\mathbf{x}^k \pm \mathbf{y}^k) = \lim \mathbf{x}^k \pm \lim \mathbf{y}^k,$$
$$\lim (\alpha \mathbf{x}^k) = \alpha \lim \mathbf{x}^k,$$
$$\lim (\alpha_k \mathbf{x}) = \mathbf{x} \lim \alpha_k \quad (k \to \infty),$$

where α and α_k are numbers. Indeed, if $\lim \mathbf{x}^k = \mathbf{x}^0$, $\lim \mathbf{y}^k = \mathbf{y}^0$, $\lim \alpha_k = \alpha_0$, then

$$| \mathbf{x}^k \pm \mathbf{y}^k - (\mathbf{x}^0 \pm \mathbf{y}^0) | \leqslant | \mathbf{x}^k - \mathbf{x}^0 | + |\mathbf{y}^k - \mathbf{y}^0 | \to 0,$$
$$| \alpha \mathbf{x}^k - \alpha \mathbf{x}^0 | = | \alpha || \mathbf{x}^k - \mathbf{x}^0 | \to 0,$$
$$| \alpha_k \mathbf{x} - \alpha_0 \mathbf{x} | = | \alpha_k - \alpha_0 || \mathbf{x} | \to 0.$$

Note that a *continuous curve* Γ can be considered in the space R. This is a set of points

$$\mathbf{x} = \mathbf{x}(t) \in R, \quad a \leqslant t \leqslant b,$$

depending on the parameter t, belonging to a closed interval $[a, b]$ and possessing the following property:

$$| \mathbf{x}(t) - \mathbf{x}(t_0) | \underset{t \to t_0}{\to} 0, \quad t, t_0 \in [a, b]. \tag{4}$$

In terms of coordinates, the points $\mathbf{x} = \mathbf{x}(t)$ of the curve Γ can be written in the following way:

$$x_j = x_j(t), \quad a \leqslant t \leqslant b \quad (j = 1, \ldots, n),$$

where $x_j(t)$ are continuous functions of t. Indeed,

$$| \mathbf{x}(t) - \mathbf{x}(t_0) | = \left(\sum_{j=1}^{n} [x_j(t) - x_j(t_0)]^2 \right)^{1/2}.$$

If the left-hand side of this equality tends to zero as $t \to t_0$, then, obviously, for any $j = 1, \ldots, n$

$$x_j(t) \to x_j(t_0), \quad t \to t_0, \tag{5}$$

and conversely, if properties (5) are fulfilled for any $j = 1, \ldots, n$, then property (4) takes place.

Sec. 8.2. Open Set

Let us choose an arbitrary point $\mathbf{x}^0 = (x_1^0, \ldots, x_n^0)$ in the n-dimensional space $R_n = R$. By a *closed sphere* (or a *closed ball*) of radius $r > 0$ with centre at the point $\mathbf{x}^0$ is meant the set of points $\mathbf{x} = (x_1, \ldots, x_n) \in R$, satisfying the inequality

$$|\mathbf{x}-\mathbf{x}^0| = [\sum_1^n (x_j - x_j^0)^2]^{1/2} \leqslant r.$$

By an *open sphere* (or an *open ball*) of radius r with centre at $\mathbf{x}^0$ we shall mean the set of points $\mathbf{x}$ for which the strict inequality

$$|\mathbf{x} - \mathbf{x}^0| < r$$

is fulfilled.

A *closed rectangle* in R (or, which is the same, a closed rectangular parallelepiped in R) is a set of points $\mathbf{x} \in R$ whose coordinates satisfy inequalities $a_j \leqslant x_j \leqslant b_j$ ($a_j < b_j$, $j = 1, \ldots, n$). In the case $n = 3$ this is a real rectangular parallelepiped with faces parallel to the axes of the rectangular Cartesian coordinates (x_1, x_2, x_3). We can also define an *open rectangle in* R as a set of points $\mathbf{x} = (x_1, \ldots, x_n)$ satisfying strict inequalities $a_j < x_j < b_j$ ($j = 1, \ldots, n$).

A set of points $\mathbf{x}$ whose coordinates satisfy inequalities $|x_j - x_j^0| \leqslant a$ ($j = 1, \ldots, n$), where $a > 0$ is a given number is naturally called a *cube* (or a *closed cube*) in R with centre at the point $\mathbf{x}^0$ and side of length $2a$. Of course, for $n = 3$ this is a cube with faces parallel to the axes of the Cartesian rectangular coordinate system. Finally, an *open cube* is defined by means of inequalities $|x_j - x_j^0| < a$ ($j = 1, \ldots, n$).

Every point $\mathbf{x}$ of a ball $|\mathbf{x} - \mathbf{x}^0| < r$ of radius r with centre at $\mathbf{x}^0$ has the coordinates x_j satisfying the inequality

$$|x_j - x_j^0| \leqslant \left(\sum_{i=1}^n (x_i - x_i^0)^2\right)^{1/2} < r.$$

This shows that the point $\mathbf{x}$ belongs to the cube

$$|x_j - x_j^0| < r \quad (j = 1, \ldots, n).$$

Thus, the *cube with side* $2r$ *and centre at* $\mathbf{x}^0$ *entirely contains the ball of radius* r *and centre at the same point.* Figure 95 explains this fact for the two-dimensional case. In its turn, the *ball of radius* $a\sqrt{n}$ *and centre at* $\mathbf{x}^0$ *entirely contains the cube* $|x_j - x_j^0| < a$ $(j = 1, \ldots, n)$

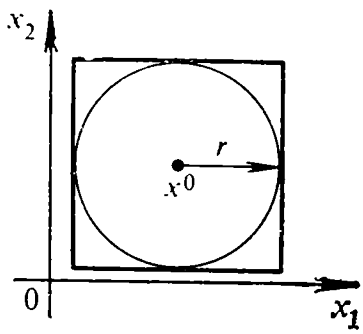

Fig. 95

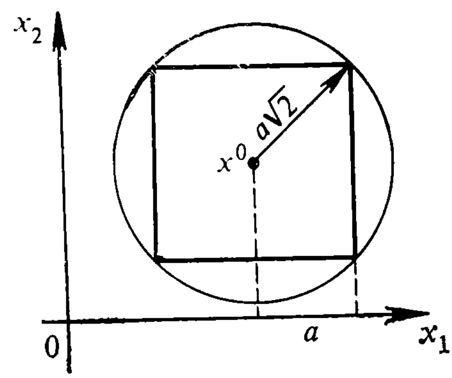

Fig. 96

with side $2a$ *and centre at the same point*, since the coordinates of the points belonging to this cube satisfy the inequality

$$\left(\sum_{j=1}^{n} (x_j - x_j^0)^2\right)^{1/2} < \left(\sum_{j=1}^{n} a^2\right)^{1/2} = a\sqrt{n}.$$

For the two-dimensional case this fact is explained in Fig. 96.

We have considered open balls and cubes; the same argument applies, of course, to closed balls and cubes.

Let us consider an arbitrary set E of points $\mathbf{x} \in R$. By definition, $\mathbf{x}^0$ is called an *interior point* of the set E if there is an open ball with centre at that point entirely contained in E. Here "ball" may be replaced by "cube", since every ball contains a cube with the same centre, and vice versa.

A set is said to be *open* if all its points are interior. This definition can be formulated in other words: *a set* E *is open if the fact that a given point belongs to it implies that it is an interior point of* E. Hence, it is clear that an *empty set is an open set.*

An open ball

$$|\mathbf{x}-\mathbf{x}^0| < r \tag{1}$$

is also an open set. Indeed, let $\mathbf{y}$ be a point belonging to the given ball, i.e. $|\mathbf{y}-\mathbf{x}^0| = \rho < r$, and let $\mathbf{x}$ be an arbitrary point belonging to the ball

$$|\mathbf{x}-\mathbf{y}| < \varepsilon \quad (\varepsilon < r-\rho). \tag{2}$$

For it, the following relation is fulfilled:

$$|\mathbf{x}-\mathbf{x}^0| = |\mathbf{x}-\mathbf{y}+\mathbf{y}-\mathbf{x}^0| \leqslant |\mathbf{x}-\mathbf{y}| + |\mathbf{y}-\mathbf{x}^0| < \varepsilon+\rho < r,$$

which shows that ball (2) is contained in ball (1). Consequently, $\mathbf{y}$ is an interior point and ball (1) is an open set.

Let the reader prove that an *open rectangle* and, in particular, an *open cube are open sets.*

By a *neighbourhood of a point* $\mathbf{x}^0 \in R$ is meant an arbitrary open set containing that point. It is obvious that the *intersection of two neighbourhoods of* $\mathbf{x}^0$ *is again a neighbourhood of* $\mathbf{x}^0$.

The aforesaid allows us to restate the definition of an interior point of a set E: $\mathbf{x}^0$ *is an interior point of* E *if there is a neighbourhood of* $\mathbf{x}^0$ *belonging to* E. Indeed, if $\mathbf{x}^0$ is an interior point in the sense of the first definition, then there is an open ball with centre at $\mathbf{x}^0$ belonging to E: such a ball is a neighbourhood of $\mathbf{x}^0$ contained in E. Conversely, if $\mathbf{x}^0$ is an interior point in the sense of the second definition, then there exists a neighbourhood of $\mathbf{x}^0$, belonging to E, which contains an open ball with centre at $\mathbf{x}^0$, since this neighbourhood is an open set.

In our present book we shall deal with many examples of open sets defined in a strict mathematical manner, but now, using solely the geometrical intuition, we can roughly say that if the boundary of a geometrical object is deleted from it, then the resultant object is an open set.

In the sections that follow we shall consider functions $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ of n variables $x_1, \ldots, x_n$ or, which is the same, of the point (n-tuple) $\mathbf{x} = (x_1, \ldots, x_n)$ defined on open sets in the n-dimensional space R_n.

A set E is said to be (*arcwise*) *connected* if its any two points $\mathbf{x}'$ and $\mathbf{x}''$ can be joined with a continuous curve

entirely lying within E, i.e. if there exists a continuous vector function $\mathbf{x} = \mathbf{x}(t)$, $0 \leqslant t \leqslant 1$, such that $\mathbf{x}(0) = \mathbf{x}'$, $\mathbf{x}(1) = \mathbf{x}''$, $\mathbf{x}(t) \in E$ (see the preceding section).

By a *line segment* $\overline{\mathbf{x}'\mathbf{x}''}$ is meant the parametrically represented curve $\mathbf{x}(t) = t\mathbf{x}' + (1 - t)\mathbf{x}''$, $t \in [0, 1]$, which is, obviously, a continuous curve connecting $\mathbf{x}'$ and $\mathbf{x}''$.

An open connected set is called the *domain*.

Sec. 8.3. Limit of a Function

The notion of the limit of a function of one variable was discussed in Sec. 3.2. Here this notion is generalized for functions of many variables.

A *function* $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ *is said to have a limit at a point* $\mathbf{x}^0 = (x_1^0, \ldots, x_n^0)$ equal to a number A and designated as

$$\lim_{x \to x^0} f(\mathbf{x}) = \lim_{\substack{x_j \to x_j^0 \\ (j=1, \ldots, n)}} f(x_1, \ldots, x_n) = A \tag{1}$$

(also written $f(\mathbf{x}) \to A$ $(\mathbf{x} \to \mathbf{x}^0)$), if the function is defined in a neighbourhood of the point $\mathbf{x}^0$ except, possibly, at that point itself and if for any sequence of points $\mathbf{x}^k$ $(k = 1, 2, \ldots)$ belonging to this neighbourhood, different from $\mathbf{x}^0$ and tending to $\mathbf{x}^0$ (see Sec. 8.1), there exists the limit

$$\lim_{\substack{|\mathbf{x}^k - \mathbf{x}^0| \to 0 \\ \mathbf{x}^k \neq \mathbf{x}^0}} f(\mathbf{x}^k) = A, \tag{2}$$

Another equivalent definition reads: *a function f has a limit at a point* $\mathbf{x}^0$ *equal to* A if it is defined in a neighbourhood of $\mathbf{x}^0$ except, possibly, at the point $\mathbf{x}^0$ itself and if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$| f(\mathbf{x}) - A | < \varepsilon \tag{3}$$

for all $\mathbf{x}$'s satisfying the inequalities

$$0 < | \mathbf{x} - \mathbf{x}^0 | < \delta. \tag{4}$$

This latter definition is, in its turn, equivalent to the following: given any $\varepsilon > 0$, there is a neighbourhood

$U(\mathbf{x}^0)$ such that inequality (3) is fulfilled for all $\mathbf{x} \in U(\mathbf{x}^0)$, $\mathbf{x} \neq \mathbf{x}^0$.

Now we state Cauchy's criterion for the existence of limit (which is proved as in the one-dimensional case; see Sec. 3.2, Theorem 5).

For a function f to have a (finite) limit at a point $\mathbf{x}^0$, it is necessary and sufficient that for any $\varepsilon > 0$ there exists a neighbourhood $U(\mathbf{x}^0)$ of the point $\mathbf{x}^0$ (in particular, a cube or a ball with centre at $\mathbf{x}^0$) *such that the inequality*

$$|f(\mathbf{x}) - f(\mathbf{x}')| < \varepsilon$$

holds for all the points $\mathbf{x}, \mathbf{x}' \in U(\mathbf{x}^0)$, distinct from $\mathbf{x}^0$.

If a number A is the limit of $f(\mathbf{x})$ at a point $\mathbf{x}^0$, then obviously, A serves as the limit of the function $f(\mathbf{x}^0 + \mathbf{h})$ of the argument $\mathbf{h}$ at the point $\mathbf{h} = 0$!, that is,

$$\lim_{\mathbf{h}\to 0} f(\mathbf{x}^0 + \mathbf{h}) = A,$$

and vice versa.

Let us consider a function f defined at all points of a neighbourhood of a point $\mathbf{x}^0$ except, possibly, at the point $\mathbf{x}^0$ itself, an arbitrary vector $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ of unit length ($|\boldsymbol{\omega}| = 1$) and a scalar $t \geqslant 0$. The collection of the points of the form $\mathbf{x}^0 + t\boldsymbol{\omega}$ ($0 \leqslant t$) is spoken of as a *ray issued from $\mathbf{x}^0$ in the direction of the vector $\boldsymbol{\omega}$*. For every $\boldsymbol{\omega}$ we can take the function

$$f(\mathbf{x}^0 + t\boldsymbol{\omega}) = f(x_1^0 + t\omega_1, \ldots, x_n^0 + t\omega_n) \quad (0 < t < \delta_\omega)$$

dependent on the scalar variable t, the number δ_ω being dependent on $\boldsymbol{\omega}$. The limit of this function (of one variable t) as $t \to 0$, provided it exists, will be called the *limit of f at the point $\mathbf{x}^0$ in the direction of the vector $\boldsymbol{\omega}$*:

$$\lim_{t\to 0,\ t>0} f(\mathbf{x}^0 + t\boldsymbol{\omega}) = \lim_{t\to 0,\ t>0} f(x_1^0 + t\omega_1, \ldots, x_n^0 + t\omega_n).$$

Example 1. Let us consider the two functions

$$f(x_1, x_2) = \frac{x_1^3 + x_2^3}{x_1^2 + x_2^2} \quad \text{and} \quad \varphi(x_1, x_2) = \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2}.$$

They are defined throughout the plane (x_1, x_2), except at the point $\mathbf{x}^0 = (0, 0)$. We have (taking into considera-

tion that $|x_1|^3 \leqslant (x_1^2 + x_2^2)^{3/2}$)

$$|f(x_1, x_2)| \leqslant \frac{2(x_1^2+x_2^2)^{3/2}}{x_1^2+x_2^2} = 2(x_1^2+x_2^2)^{1/2}$$
$$= 2|\mathbf{x}| = 2|\mathbf{x}-\mathbf{x}^0|,$$

whence

$$\lim_{x_1, x_2 \to 0} f(x_1, x_2) = 0$$

(for $\varepsilon > 0$ we set $\delta = \varepsilon/2$; then $|f(x_1, x_2)| < \varepsilon$, provided $|\mathbf{x}| < \delta$).

Further, assuming that k is an arbitrary constant number, we have $x_2 = kx_1$ and

$$\varphi(x_1, kx_1) = \frac{1-k^2}{1+k^2},$$

whence it follows that the limits of φ at (0, 0) in different directions are, generally, different from each other, and therefore the function φ has no limit at (0, 0).

Example 2. Let us consider in R_2 the function

$$f(x, y) = x^2y/(x^4 + y^2), \quad x^4 + y^2 \neq 0.$$

At the point (0, 0) on any straight line $y = kx$ passing through the origin, the given function has a limit equal to zero:

$$f(x, kx) = kx^3/(x^4 + k^2x^2)$$
$$= kx/(x^2 + k^2) \to 0 \quad \text{as} \quad x \to 0.$$

But this function has no limit at the point (0, 0), since for $y = x^2$

$$f(x, x^2) = x^4/(x^4 + x^4) = 1/2 \text{ and } \lim_{\substack{x\to 0\\ y=x^2}} f(x, x^2) = 1/2.$$

We shall write $\lim\limits_{\mathbf{x}\to\mathbf{x}^0} f(\mathbf{x}) = \infty$ if the function f is defined on a neighbourhood of $\mathbf{x}^0$ except, possibly, at $\mathbf{x}^0$ and if, given any $N > 0$, there is $\delta > 0$ such that $|f(\mathbf{x})| > N$, whenever $0 < |\mathbf{x} - \mathbf{x}^0| < \delta$.

We can also introduce the notion of the limit of f for $\mathbf{x} \to \infty$:

$$\lim_{\mathbf{x}\to\infty} f(\mathbf{x}) = A. \tag{5}$$

For instance, if A is a finite number, then equality (5) should be understood in the sense that for any $\varepsilon > 0$, there exists $N > 0$ such that the function f is defined at all points $\mathbf{x}$ for which $|\mathbf{x}| > N$ and the inequality $|f(\mathbf{x}) - A| < \varepsilon$ is fulfilled for such $\mathbf{x}$'s.

There hold the equalities

$$\lim_{\mathbf{x}\to\mathbf{x}^0} (f(\mathbf{x}) \pm \varphi(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{x}^0} f(\mathbf{x}) \pm \lim_{\mathbf{x}\to\mathbf{x}_0} \varphi(\mathbf{x}), \tag{6}$$

$$\lim_{\mathbf{x}\to\mathbf{x}^0} (f(\mathbf{x})\cdot\varphi(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{x}^0} f(\mathbf{x}) \lim_{\mathbf{x}\to\mathbf{x}_0} \varphi(\mathbf{x}), \tag{7}$$

$$\lim_{\mathbf{x}\to\mathbf{x}^0} \frac{f(\mathbf{x})}{\varphi(\mathbf{x})} = \frac{\lim\limits_{\mathbf{x}\to\mathbf{x}^0} f(\mathbf{x})}{\lim\limits_{\mathbf{x}\to\mathbf{x}^0} \varphi(\mathbf{x})} \quad (\lim_{\mathbf{x}\to\mathbf{x}^0} \varphi(\mathbf{x}) \neq 0), \tag{8}$$

where, in particular, there can be $\mathbf{x}^0 = \infty$. As usual, these relations mean that the (finite) limits on their left-hand sides exist if the limits of f and φ exist. As an instance, let us prove (7).

Suppose that $\mathbf{x}^k \to \mathbf{x}^0$ $(\mathbf{x}^k \neq \mathbf{x}^0)$; then

$$\lim_{\mathbf{x}^k\to\mathbf{x}^0} (f(\mathbf{x}^k)\,\varphi(\mathbf{x}^k)) = \lim_{\mathbf{x}^k\to\mathbf{x}^0} f(\mathbf{x}^k)\cdot \lim_{\mathbf{x}^k\to\mathbf{x}^0} \varphi(\mathbf{x}^k)$$

$$= \lim_{\mathbf{x}\to\mathbf{x}^0} (f(\mathbf{x}) \lim_{\mathbf{x}\to\mathbf{x}^0} \varphi(\mathbf{x}). \tag{9}$$

Hence, the limit on the left-hand side of (9) exists and is equal to the right-hand member of (9), and, since the sequence $\{\mathbf{x}^k\}$ has been taken quite arbitrarily, this limit is equal to the limit of the function $f(\mathbf{x})\,\varphi(\mathbf{x})$ at the point $\mathbf{x}^0$.

Theorem 1. *If a function f has a nonzero limit at a point $\mathbf{x}^0$:*

$$\lim_{\mathbf{x}\to\mathbf{x}0} f(\mathbf{x}) = A \neq 0,$$

then there exists $\delta > 0$ such that for all $\mathbf{x}$'s satisfying the inequalities

$$0 < |\mathbf{x} - \mathbf{x}^0| < \delta, \tag{10}$$

the given function satisfies the inequality

$$|f(\mathbf{x})| > |A|/2. \tag{11}$$

Moreover, it retains there the sign of the number A.

Indeed, setting $\varepsilon = |A|/2$ we find $\delta > 0$ such that the inequality

$$|f(\mathbf{x}) - A| < |A|/2. \tag{12}$$

is fulfilled for all $\mathbf{x}$'s satisfying inequalities (10). Therefore, we have

$$|A|/2 > |A - f(\mathbf{x})| \geqslant |A| - |f(\mathbf{x})|,$$

for such $\mathbf{x}$'s, i.e. inequality (11) holds. For the indicated $\mathbf{x}$'s it follows from (12) that

$$A - \frac{|A|}{2} < f(\mathbf{x}) < A + \frac{|A|}{2},$$

whence $A/2 < f(\mathbf{x})$ for $A > 0$ and $f(\mathbf{x}) < A/2$ for $A < 0$.

Remark. In Sec. 8.12 we shall state a more general definition of the limit for a function defined on an arbitrary set.

Sec. 8.4. Continuous Functions

A function $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ is said to be *continuous at a point* $\mathbf{x}^0 = (x_1^0, \ldots, x_n^0)$ if it is defined in a neighbourhood of $\mathbf{x}^0$ including the point $\mathbf{x}^0$ itself and if the limit of the function at the point $\mathbf{x}^0$ exists and is equal to the value of $f(\mathbf{x})$ at $\mathbf{x}^0$:

$$\lim_{\mathbf{x} \to \mathbf{x}^0} f(\mathbf{x}) = f(\mathbf{x}^0). \tag{1}$$

The condition of the continuity of f at $\mathbf{x}^0$ can be written in the equivalent form

$$\lim_{\mathbf{h} \to \mathbf{o}} f(\mathbf{x}^0 + \mathbf{h}) = f(\mathbf{x}^0), \tag{1'}$$

i.e. the function $f(\mathbf{x})$ is continuous at the point $\mathbf{x}^0$ if the function $f(\mathbf{x}^0 + \mathbf{h})$ of the argument $\mathbf{h}$ is continuous at the point $\mathbf{h} = \mathbf{o}$.

We can also introduce the notion of the increment of f at the point $\mathbf{x}^0$ corresponding to the increment $\mathbf{h} = (h_1, \ldots, h_n)$:

$$\Delta_{\mathbf{h}} f(\mathbf{x}^0) = f(\mathbf{x}^0 + \mathbf{h}) - f(\mathbf{x}^0)$$

and restate the definition of the continuity of f at $\mathbf{x}^0$ in terms of the increments: the *function f is continuous at $\mathbf{x}^0$ if*

$$\lim_{\mathbf{h}\to\mathbf{o}} \Delta_{\mathbf{h}} f(\mathbf{x}^0) = \lim_{h_1, \ldots, h_n \to 0} [f(x_1^0+h_1, \ldots, x_n^0+h_n) - f(x_1^0, \ldots, x_n^0)] = 0. \qquad (1'')$$

From formulas (6)-(8) of the preceding section we directly deduce the following theorem.

Theorem 1. *The sum, the difference, the product, and the quotient of two functions $f(\mathbf{x})$ and $\varphi(\mathbf{x})$, continuous at a point $\mathbf{x}^0$, are continuous functions at that point (provided that $\varphi(\mathbf{x}^0) \neq 0$ in the case of the quotient).*

A constant c can be regarded as the function $f(\mathbf{x}) = c$ of $\mathbf{x} = (x_1, \ldots, x_n)$. It is continuous for any $\mathbf{x}$, since

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = c - c = 0 \to 0 \quad (\mathbf{h} \to \mathbf{o}).$$

A more complicated function is $f_j(\mathbf{x}) = x_j$ ($j = 1, \ldots, \ldots, n$), where the index j can be equal to one of the numbers $1, \ldots, n$. It is also continuous (as a function of $\mathbf{x} = (x_1, \ldots, x_n)$!). Indeed, let $\mathbf{h} = (h_1, \ldots, h_n)$; then

$$|f_j(\mathbf{x}+\mathbf{h}) - f_j(\mathbf{x})| = |(x_j+h_j) - x_j| = |h_j| \leqslant |\mathbf{h}| \to 0, \quad (\mathbf{h} \to \mathbf{o}).$$

A finite number of operations of addition, subtraction and multiplication performed at the functions x_j and on constants leads to a function called a *polynomial of* $\mathbf{x}$ (or a *polynomial in the variables* $x_1, \ldots, x_n$). On the basis of the above-stated properties, we conclude that the *polynomials are continuous functions in R_n* (for all $\mathbf{x} \in R_n$). The ratio P/Q of two polynomials is called a *rational function* which is, obviously, continuous throughout R_n except at the points $\mathbf{x}$ at which $Q(\mathbf{x}) = 0$.

The function

$$P(\mathbf{x}) = x_1^3 - x_2^3 + x_1^2 x_3 + 2x_1^2 x_2 - 3x_3^2 + 4$$

is an example of a polynomial of the third degree in the variables x_1, x_2, and x_3.

There holds the following general theorem.

Theorem 2. *Let $f(\mathbf{x}) = f(x_1, \ldots, x_m)$ be a continuous function at a point $\mathbf{x}^0 = (x_1^0, \ldots, x_m^0)$ in the space R_m*

(of points $\mathbf{x}$*) and* $\varphi_j(\mathbf{u}) = \varphi_j(u_1, \ldots, u_n)$ *be continuous functions at a point* $\mathbf{u}^0 = (u_1^0, \ldots, u_n^0)$ *in the space* R_n *(of points* $\mathbf{u}$*). Besides, let* $\varphi_j(\mathbf{u}^0) = x_j^0$ $(j = 1, \ldots, m)$*. Then the function*

$$F(\mathbf{u}) = f(\varphi_1(\mathbf{u}), \varphi_2(\mathbf{u}), \ldots, \varphi_m(\mathbf{u}))$$

is continuous (with respect to $\mathbf{u}$*) at the point* $\mathbf{u}^0$.

Proof. Since f is continuous at $\mathbf{x}^0$, for any $\varepsilon > 0$ there is $\delta > 0$ such that f is defined for all $\mathbf{x}$ satisfying the condition $|x_j - x_j^0| < \delta$ $(j = 1, \ldots, m)$ and such that the inequality $|f(\mathbf{x}) - f(\mathbf{x}^0)| < \varepsilon$ holds for these $\mathbf{x}$'s. Further, since the functions φ_j are continuous at the point $\mathbf{u}^0$ of the space R_n there is $\eta > 0$ such that for the points $\mathbf{u} \in R_n$ belonging to the ball $|\mathbf{u} - \mathbf{u}^0| < \eta$ there hold the inequalities

$$|\varphi_j(\mathbf{u}) - \varphi_j(\mathbf{u}^0)| < \delta \quad (j = 1, \ldots, m).$$

Then the inequality

$$|F(\mathbf{u}) - F(\mathbf{u}^0)| = |f(\varphi_1(\mathbf{u}), \ldots, \varphi_m(\mathbf{u})) - f(\varphi_1(\mathbf{u}^0), \ldots, \varphi_m(\mathbf{u}^0))| < \varepsilon$$

also holds, which completes the proof of the theorem.

A function will be spoken of as an *elementary function* of the variables $x_1, \ldots, x_n$ if it can be obtained from these variables and from constants c with the aid of a finite number of operations of addition, subtraction, multiplication, division, and operations φ, where φ symbolizes elementary functions of one variable (see Sec. 3.8). The functions

$$(1)\ \sin \ln \sqrt{1 + x^2 + y^2} = f_1, \quad (2)\ \sin^2 x + \cos^3(x + y) = f_2,$$

$$(3)\ \ln \frac{x-y}{x+y} = f_3$$

can serve as examples of elementary functions of several variables.

Using Theorems 1-2, we can readily check that the functions f_1 and f_2 are continuous throughout the xy-plane, while the function f_3 is, obviously, defined and continuous at the points (x, y) for which the fraction $(x - y)/(x + y)$ is finite and positive.

From Theorem 1 in Sec. 8.3 and from the definition of the continuity of a function at a point follows directly

Theorem 3. *If a function* $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ *is continuous at a point* $\mathbf{x}^0$ *and assumes a nonzero value at that point, then it retains sign in a neighbourhood of* $\mathbf{x}^0$ *which coincides with the sign of* $f(\mathbf{x}^0)$.

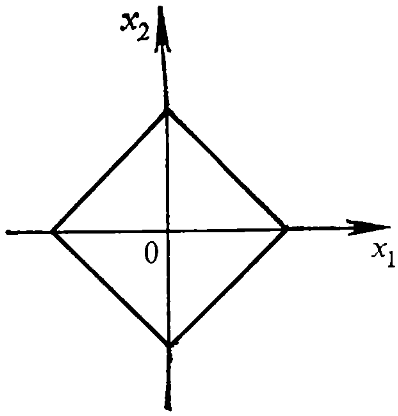

Fig. 97

Corollary. *Let a function* $f(\mathbf{x})$ *be defined and continuous throughout* R_n *(i.e. at all the points of* R_n*). Then the set* G *of the points* $\mathbf{x}$ *at which the function satisfies the inequality* $f(\mathbf{x}) > c$ *(or* $f(\mathbf{x}) < c$*), where* c *is an arbitrary constant, is an open set.*

Indeed, the function $F(\mathbf{x}) = f(\mathbf{x}) - c$ is continuous on R_n, and the set of all points $\mathbf{x}$, where $F(\mathbf{x}) > 0$, coincides with G. Let $\mathbf{x}^0 \in G$, then there is a ball

$$|\mathbf{x} - \mathbf{x}^0| < \delta,$$

on which $F(\mathbf{x}) > 0$, this means that the ball belongs to G and, therefore, $\mathbf{x}^0 \in G$ is an interior point of G.

The case $f(\mathbf{x}) < c$ is proved in a similar way.

Example. The functions

$$f_1(\mathbf{x}) = \sum_1^n \frac{x_k^2}{a_k} \quad (a_k > 0);$$

$$f_2(\mathbf{x}) = \sum_1^n |x_k|,$$

are defined and continuous in R_n.

In such a case the sets of points $\mathbf{x}$ for which the inequalities $f_i(\mathbf{x}) < c$ $(i = 1, 2)$ hold are open. The first of them is the interior of an ellipsoid in the n-dimensional space; the second (for $n = 2$) is the interior of a square, shown in Fig. 97.

The inequalities $f_i(\mathbf{x}) > c > 0$ specify the exteriors of these figures.

Sec. 8.5. Partial Derivatives and Directional Derivative

In this section we shall consider functions f defined on an arbitrary open set $G \subset R_n$.

By the *increment of* f *at a point* $\mathbf{x} = (x_1, \ldots, x_n) \in G$ *with respect to the variable* x_j *with step* h *we shall mean*

the difference

$$\Delta_{x_j h} f(\mathbf{x}) = f(x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_n) - f(x_1, \ldots, x_n),$$

where h is a real number which is sufficiently small so that this difference makes sense.

The partial derivative of f with respect to x_j at the point $\mathbf{x}$ is defined as the limit

$$f'_{x_j} = \frac{\partial f}{\partial x_j} = \frac{\partial f(\mathbf{x})}{\partial x_j} = \lim_{h \to 0} \frac{\Delta_{x_j h} f(\mathbf{x})}{h} \quad (j = 1, \ldots, n),$$

provided that it exists. The partial derivative $\frac{\partial f(\mathbf{x})}{\partial x_j}$ is nothing but the ordinary derivative of the function $f(x_1, \ldots, x_n)$ regarded as a function of the variable x_j alone for fixed $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$.

A function $z = f(x, y)$ of two variables is represented geometrically by the locus of points $(x, y, f(x, y))$, where $(x, y) \in G$, that is, by a surface in the three-dimensional space in which rectangular Cartesian coordinates (x, y, z) are introduced. It is evident that the *derivative $f'_x(x_0, y_0)$* (provided it exists) *is equal to the tangent of the angle of inclination (to the x-axis) of the tangent line to the section of that surface by the plane $y = y_0$ drawn through the point with abscissa x_0.*

If the function $u = f(x_1, \ldots, x_n)$ has partial derivatives $\frac{\partial f}{\partial x_k}$ $(k = 1, \ldots, n)$ at all points $\mathbf{x} = (x_1, \ldots, x_n) \in G$, then these derivatives can be regarded as new functions defined on the set G.

Therefore we have the right to consider the question concerning the existence of partial derivatives of these functions with respect to some variable at the point $\mathbf{x}$.

If the function $\frac{\partial f}{\partial x_k}$ has a partial derivative again with respect to x_k, then the latter is called the *partial derivative of the second order (the second partial derivative) of the function $f(\mathbf{x})$ with respect to the variable x_k* and is denoted $\frac{\partial^2 f}{\partial x_k^2}$. Thus, by definition,

$$\frac{\partial^2 f}{\partial x_k^2} = \frac{\partial}{\partial x_k}\left\{\frac{\partial f}{\partial x_k}\right\}.$$

If there exists a partial derivative of the function $\frac{\partial f}{\partial x_k}$ with respect to some other variable x_i $(i \neq k)$, then this derivative is termed a *mixed partial derivative of the second order and denoted by the symbol*

$$\frac{\partial^2 f}{\partial x_i \, \partial x_k} = \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_k} \right).$$

Hence, for a function of two variables $f(x, y)$ four derivatives of the second order are possible:

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x \, \partial y}, \quad \frac{\partial^2 f}{\partial y \, \partial x}, \quad \frac{\partial^2 f}{\partial y^2}.$$

If derivatives of the second order (all or one of them) exist for all $\mathbf{x} \in G$, then there can arise a question concerning the existence of partial derivatives of the third order.

In general, a *partial derivative of the nth order* will be understood as a partial derivative of some derivative of order $n - 1$ with respect to some variable. For instance,

$$\frac{\partial}{\partial x} \left(\frac{\partial^3 f}{\partial x^2 \partial y} \right) = \frac{\partial^4 f}{\partial x^3 \, \partial y}.$$

The derivatives $\frac{\partial f}{\partial x_k}$ will be called the *partial derivatives of the first order*, and the *function f itself the partial derivative of zero order.*

The following notation will also be used for partial derivatives:

$$D_{x_1} f = \frac{\partial f}{\partial x_1}, \quad D_{x_1} D_{x_2} f$$

$$= \frac{\partial^2 f}{\partial x_1 \, \partial x_2}, \quad \ldots, \quad \frac{\partial^2 f}{\partial x^2} = f''_{x^2}, \quad \frac{\partial^2 f}{\partial x \, \partial y} = f''_{xy}.$$

If $\mathbf{r} = (r_1, \ldots, r_n)$ is a vector with nonnegative integral coordinates, then we write

$$D^{\mathbf{r}} f = D_{x_1}^{r_1} \ldots D_{x_n}^{r_n} f = \frac{\partial^{r_1 + \ldots + r_n} f}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}}.$$

Example. Find $\frac{\partial^3 f}{\partial x\,\partial y^2}$ of the function $f(x, y) = x^2 + \sin xy$. We have

$$\frac{\partial f}{\partial y} = x\cos xy, \qquad \frac{\partial^2 f}{\partial y^2} = -x^2 \sin xy,$$

$$\frac{\partial^3 f}{\partial x\,\partial y^2} = -2x\sin xy - x^2 y\cos xy.$$

There naturally arises a question whether partial derivatives will be equal to one another if they are taken with respect to the same variables one and the same number of times but in different orders.

For instance, are

$$\frac{\partial^4 f}{\partial x\,\partial y^2\,\partial x} \quad \text{and} \quad \frac{\partial^4 f}{\partial x^2\,\partial y^2}$$

equal to each other?

In the general case the answer to this question is negative. But there takes place the following theorem which is formulated for a function of two variables.

Theorem (on Mixed Derivatives). *Let a function $u = f(x, y)$ be defined together with its partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x\,\partial y}$, $\frac{\partial^2 f}{\partial y\,\partial x}$ in a certain neighbourhood of the point $P_0 = (x_0, y_0)$, and $\frac{\partial^2 f}{\partial x\,\partial y}$, $\frac{\partial^2 f}{\partial y\,\partial x}$ be continuous at the point P_0, then $\frac{\partial^2 f(P_0)}{\partial x\,\partial y} = \frac{\partial^2 f(P_0)}{\partial y\,\partial x}$, i.e. in this case the result of differentiation is independent of the order of differentiation.*

We are not going to prove this theorem and confine ourselves to the following remarks.

Remark 1. Using the method of induction, it is easy to extend this theorem to any continuous mixed partial derivatives which differ from one another by the order of differentiation. For instance,

$$\frac{\partial^3 f}{\partial x\,\partial y^2} = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]\right\} = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]\right\}$$

$$\equiv \frac{\partial^3 f}{\partial y\,\partial x\,\partial y} = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]\right\} \equiv \frac{\partial^3 f}{\partial y^2\,\partial x}.$$

Remark 2. If the continuity condition is absent, then mixed derivatives can be different at the point P_0. Let us consider the function

$$f(x, y) = \frac{x^3 y}{x^2+y^2} \text{ if } x^2+y^2 \neq 0 \text{ and } f(0, 0) = 0.$$

It is easy to compute that

$$f'_x(x, y) = y\frac{x^4+3x^2y^2}{(x^2+y^2)^2} \text{ for } x^2+y^2 \neq 0 \text{ and } f'_x(0, 0) = \lim_{x\to 0} \frac{f(x, 0)-f(0, 0)}{x} = 0,$$

$$f'_y(x, y) = x^3 \frac{x^2-y^2}{(x^2+y^2)^2} \text{ for } x^2+y^2 \neq 0 \text{ and } f'_y(0, 0) = 0.$$

Further, by definition,

$$f''_{xy}(0, 0) = \lim_{x\to 0} \frac{f'_y(x, 0)-f'_y(0, 0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1,$$

$$f''_{yx}(0, 0) = \lim_{y\to 0} \frac{f'_x(0, y)-f'_x(0, 0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0,$$

i. e.

$$f''_{xy}(0, 0) \neq f''_{yx}(0, 0).$$

Note that the partial derivatives f''_{xy} and f''_{yx} are discontinuous at the point (0, 0); for instance $f''_{yx}(x, y) = \frac{x^6+6x^4y^2-3x^2y^4}{(x^2+y^2)^3}$ for $x^2+y^2 \neq 0$, whence it is seen that

$$\lim_{\substack{x\to 0 \\ x=y}} f''_{yx}(x, y) = \frac{1}{2} \neq f''_{yx}(0, 0) = 0.$$

Another important notion connected with differentiation is the *directional derivative* (it is not used in the case of one independent variable).

Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ be an arbitrary unit vector. The (*directional*) *derivative of a function f at a point* $\mathbf{x}$ *in the direction of* $\boldsymbol{\omega}$ (along $\boldsymbol{\omega}$) is defined as the limit

$$\frac{\partial f}{\partial \omega} = \lim_{\substack{t\to 0 \\ t>0}} \frac{f(\mathbf{x}+t\boldsymbol{\omega})-f(\mathbf{x})}{t}$$

(provided it exists). It should be stressed that in the computation of this limit it is assumed that t tends to zero taking on *positive* values; therefore we can also say that $\frac{\partial f(x)}{\partial \omega}$ *is the right-hand derivative of the function* $f(\mathbf{x} + t\boldsymbol{\omega})$ *with respect to* t *at the point* $t = 0$.

By analogy with functions of one variable, we can speak of the right-hand and left-hand partial derivatives with respect to x_j. It should also be taken into account that the *directional derivative along the positive x_j-axis coincides with the right-hand partial derivative with respect to x_j, while the directional derivative along the negative x_j-axis has the sign opposite to the sign of the left-hand derivative with respect to x_j.*

Sec. 8.6. Differentiable Functions

For the sake of simplicity, we shall consider the three-dimensional case; in the general case of dimension n the argument is quite analogous. The case $n = 1$ was considered separately in Sec. 4.7.

Let a function $u = f(x, y, z)$ be defined in an open set $G \subset R_3$ and let it possess continuous partial derivatives of the first order at a point $(x, y, z) \in G$. This implies automatically that these partial derivatives also exist in some neighbourhood of (x, y, z), although they may not be continuous at points different from (x, y, z). Let us consider the increment of f at (x, y, z) corresponding to an increment $(\Delta x, \Delta y, \Delta z)$, where $|\Delta x|$, $|\Delta y|$, $|\Delta z|$ are less than δ and δ is sufficiently small so that the point $(x + \Delta x, y + \Delta y, z + \Delta z)$ does not fall outside the indicated neighbourhood. Then there hold the following equalities (see the explanations below)

$$\begin{aligned}
\Delta u &= f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z) && (1)\\
&= f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y + \Delta y, z + \Delta z) && (2)\\
&\quad + f(x, y + \Delta y, z + \Delta z) - f(x, y, z + \Delta z) && (3)\\
&\quad + f(x, y, z + \Delta z) - f(x, y, z) && (4)\\
&= f'_x(x + \theta_1 \Delta x, y + \Delta y, z + \Delta z)\,\Delta x
\end{aligned}$$

$$+ f'_y(x, y+\theta_2\Delta y, z+\Delta z)\,\Delta y + f'_z(x, y, z + \theta_3\Delta z)\,\Delta z \quad (5)$$

$$= (f'_x(x, y, z) + \varepsilon_1)\,\Delta x + (f'_y(x, y, z) + \varepsilon_2)\,\Delta y + (f'_z(x, y, z) + \varepsilon_3)\,\Delta z \quad (6)$$

$$= f'_x(x, y, z)\,\Delta x + f'_y(x, y, z)\,\Delta y + f'_z(x, y, z)\,\Delta z + o(\rho) \quad (7)$$

$$(\rho \to 0),$$

$$0 < \theta_1, \theta_2, \theta_3 < 1, \ \rho = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}, \quad \varepsilon_1, \varepsilon_2, \varepsilon_3 \to 0 \quad (8)$$

$$(\rho \to 0).$$

The passage from (2) to the first term in (5) is justified as follows. By the hypothesis, the function $f(\xi, y+\Delta y, z+\Delta z)$ of the variable ξ (considered for fixed $y+\Delta y$ and $z+\Delta z$) has the derivative (with respect to ξ) on the interval $[x, x+\Delta x]$, and Lagrange's mean-value theorem is applicable to it. The second and third terms in (5) are explained similarly. The passage from (5) to (6) is purely formal; for instance, we simply set

$$f'_x(x+\theta_1\Delta x, y+\Delta y, z+\Delta z) = f'_x(x, y, z) + \varepsilon_1.$$

But the fact that $\varepsilon_1 \to 0$ as $\rho \to 0$ is not formal; it is a consequence of the assumption that f'_x is continuous at (x, y, z). Finally, the passage from (6) to (7) reduces to the assertion that there holds the relation

$$\varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y + \varepsilon_3\,\Delta z = o(\rho) \quad (\rho \to 0).$$

Indeed, since $|\Delta x|, |\Delta y|, |\Delta z| \leqslant \rho$, as $\rho \to 0$

$$|\varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y + \varepsilon_3\,\Delta z|/\rho \leqslant |\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| \to 0.$$

We have thus proved the following important theorem.

Theorem 1. *If a function $u = f$ has continuous first-order derivatives at a point (x, y, z), then its increment at this point, corresponding to a sufficiently small increment $(\Delta x, \Delta y, \Delta z)$, is representable in the form*

$$\Delta u = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z + o(\rho) \quad (\rho \to 0), \quad (9)$$

$$\rho = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2},$$

where the values of the partial derivatives are taken at the point (x, y, z).

Since the values of the partial derivatives on the right-hand side of (9) are independent of Δx, Δy, Δz, the conditions of Theorem 1 imply that the increment of f at (x, y, z), corresponding to the increment $(\Delta x, \Delta y, \Delta z)$, can be written in the form

$$\Delta u = A\ \Delta x + B\ \Delta y + C\ \Delta z + o(\rho) \quad (\rho \to 0), \tag{10}$$

where the numbers A, B, and C are independent of Δx, Δy, Δz.

Let us state the following definition: if the increment of a function f at a point (x, y, z) can be represented as sum (10) for sufficiently small $(\Delta x, \Delta y, \Delta z)$, where A, B, C are constant numbers independent of Δx, Δy, Δz, the function f is said to be *differentiable at the point* (x, y, z).

Thus, *the differentiability of f at (x, y, z) means that the increment Δf of the function f at this point can be written as the sum of two terms one of which is the linear function $A\Delta x + B\ \Delta y + C\ \Delta z$ of $(\Delta x, \Delta y, \Delta z)$ called the principal linear part of the increment Δf, and the other is an expression dependent on the increments Δx, Δy, Δz (which can be a complicated function of Δx, Δy, and Δz in the general case) such that when Δx, Δy, and Δz are made to tend to zero it tends to zero faster than $\rho = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$.*

It is easily seen that if a function f is differentiable at a point (x, y, z), i.e. if its increment is representable in form (10), then it has the first derivatives at that point equal respectively to A, B, C:

$$\frac{\partial f}{\partial x} = A, \quad \frac{\partial f}{\partial y} = B, \quad \frac{\partial f}{\partial z} = C. \tag{11}$$

As an instance, let us prove the first relation (11). Suppose that the increment of f at (x, y, z) is representable in form (10). Setting $\Delta x = h$, $\Delta y = \Delta z = 0$ in (10), we obtain the equality $\Delta_{xh} u = Ah + o(h)$ $(h \to 0)$. On dividing the latter by h and passing to the limit, we

arrive at

$$\lim_{h\to 0}\frac{\Delta_{xh}u}{h}=\frac{\partial f}{\partial x}=A.$$

From what has been said follows

Theorem 2. *For a function f to be differentiable at a point, it is necessary that it possess partial derivatives at that point and it is sufficient that it have continuous partial derivatives at the point.*

We remind here that for a function f of one variable to be differentiable at a point x, it is necessary and sufficient that it possess a derivative at that point.

It follows from (10) that if *a function is differentiable at a point, then it is necessarily continuous at that point.*

Example 1. The function $f(x, y, z)$, equal to zero on the coordinate planes $x = 0$, $y = 0$, $z = 0$ and equal to unity at the other points of R_3, obviously, has zero partial derivatives at the point $(0, 0, 0)$, but is discontinuous at that point and therefore cannot be differentiable at it. We see that the existence of partial derivatives of a function at a point is not sufficient for the differentiability and even for the continuity of the function at that point.

We should also stress a distinction between the one-dimensional and many-dimensional cases. In the case $n = 1$ the property of differentiability of f at x is expressed in the form of the equality $\Delta f = A\,\Delta x + o(\Delta x)$ and, consequently, if $A \neq 0$, then the second term on the right-hand side (i.e. the remainder) tends to zero as $\Delta x \to 0$ faster than the principal part. But this is no longer the case for $n > 1$, for instance, if $n = 3$, then for any three numbers A, B and C, which are not all zero, we can always make Δx, Δy, and Δz tend to zero in such a way that the equality $A\,\Delta x + B\,\Delta y + C\,\Delta z = 0$ constantly holds; but then the remainder $o(\rho)$ in (10) is even greater than the principal linear part. However, if we make Δx, Δy, and Δz tend to zero so that the proportionality relation $\Delta x : \Delta y : \Delta z = A : B : C$ is fulfilled, the principal part of the increment will be exactly of the order of ρ and the remainder will tend to zero faster than the principal part.

Example 2. The function $u = |x|(y + 1)$ is continuous at the point $(0, 0)$. But it is easily seen that $\frac{\partial u}{\partial x}$ is not existent at that point. Consequently, u is not differentiable at the point $(0, 0)$.

If a function f is differentiable at a point (x, y, z), then the principal linear part of its increment at that point is also called the *differential of f at the point (x, y, z) corresponding to the increments $(\Delta x, \Delta y, \Delta z)$ of the independent variables.* The differential is denoted as follows:

$$df = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z.$$

Its other notations will be delt with in Sec. 8.9.

Sec. 8.7. Tangent Plane. Geometrical Meaning of Differential

Let be given a surface S described by the function

$$z = f(x, y) \tag{1}$$

having continuous partial derivatives on a certain domain of the xy-plane.

The *tangent plane* to the surface S at its point $M_0 = (x_0, y_0, z_0)$, $z_0 = f(x_0, y_0)$ is defined as a plane having the equation

$$Z - z_0 = \left(\frac{\partial f}{\partial x}\right)_0 (X - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (Y - y_0), \tag{2}$$

where X, Y, Z are running coordinates, and $\left(\frac{\partial f}{\partial x}\right)_0$, $\left(\frac{\partial f}{\partial y}\right)_0$ are the values of the partial derivatives of f at the point $P_0 = (x_0, y_0)$.

Let us denote plane (2) by Π. It passes through the point M_0 of the surface S and possesses a property which distinguishes it from other planes passing through M_0.

Let $P = (x, y)$ be a point lying in the xy-plane close to $P_0 = (x_0, y_0)$ (Fig. 98). The straight line passing through P parallel to the z-axis intersects Π at the point T, and the surface S at the point M. The z-coordinate of M is equal to

$$z = f(x, y),$$

and the z-coordinate of the point T to

$$Z = f(x_0, y_0) + \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0).$$

The distance between the points M and T is equal to

$$|MT| = f(x, y) - f(x_0, y_0) - \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) - \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0). \quad (3)$$

And the distance between the points P and P_0 is

$$\rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

Since, by the hypothesis, the function f has continuous partial derivatives at the point (x_0, y_0), it is differen-

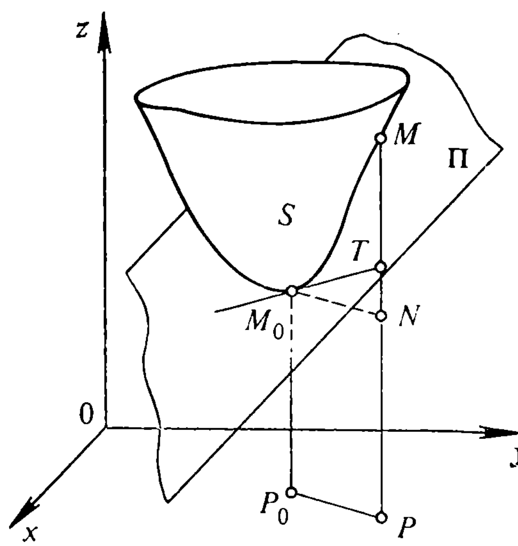

Fig. 98

tiable at that point. Therefore, the right-hand member of (3) tends to zero faster than ρ, i.e.

$$|MT| = o(\rho) \quad (\rho \to 0).$$

Hence, we have proved that a *tangent plane* Π *to the surface* S *at its point* $M_0 = (x_0, y_0, z_0)$ *passes through this point and possesses the following property: the distance from an arbitrary point* $(x, y, f(x, y))$ *on the surface* S *to* Π *measured along the z-axis is equal to* $o(\rho)$ $(\rho \to 0)$, *where* ρ *is the distance between the points* (x, y) *and* (x_0, y_0) *of the xy-plane.*

This property is characteristic for a tangent plane, since, if some plane Π' of the form

$$z - z_0 = a(x - x_0) + b(y - y_0) \qquad (z_0 = f(x_0, y_0))$$

possesses this property, i.e. if the equality

$$f(x, y) - f(x_0, y_0) - a(x - x_0) - b(y - y_0) = o(\rho) \qquad (\rho \to 0)$$

or, which is the same, the equality

$$f(x, y) - f(x_0, y_0) = a(x - x_0) + b(y - y_0) + o(\rho) \qquad (\rho \to 0)$$

is fulfilled for it, then, as we know, f is differentiable at (x_0, y_0) and

$$a = \left(\frac{\partial f}{\partial x}\right)_0, \qquad b = \left(\frac{\partial f}{\partial y}\right)_0,$$

i.e. Π' is a tangent plane to S ($\Pi' = \Pi$).

The right-hand member of equation (2) is the differential of f at the point (x_0, y_0):

$$df = \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0),$$

corresponding to the increments $(x - x_0,\ y - y_0)$, while the left-hand member of (2) is the corresponding increment of the z-coordinate of the tangent plane Π.

Thus, from the geometrical point of view, the differential of the function f at the point (x_0, y_0) for the increments $(x - x_0,\ y - y_0)$ is the increment of the z-coordinate of the point of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0) for the same increments.

Remark. If the function $z = f(x, y)$ is not differentiable at the point (x_0, y_0), although it has partial derivatives at that point, then it makes no sense to call (2) a tangent plane to the surface $z = f(x, y)$ at the indicated point, since for it the difference $f(x, y) - Z$ does not tend to zero faster than ρ as $\rho \to 0$. For instance, if the function $z = f(x, y)$ is equal to zero on the x- and y-axes and to unity at the remaining points of the xy-plane, then $f'_x(0, 0) = f'_y(0, 0) = 0$ and equation (2) becomes $Z = 0$ and the difference $f(x, y) - Z = f(x, y) - 0 = 1$ for all points (x, y) not lying on the x- and y-axes. Hence, this difference does not even tend to zero as $\rho \to 0$.

Sec. 8.8. Derivative of a Composite Function. Directional Derivative. Gradient

We shall confine ourselves to considering functions of three variables defined in an open set $G \subset R_3$. The extension of the facts established here to the n-dimensional case is carried out analogously.

Theorem 1. *Let a function*

$$u = f(x, y, z) \tag{1}$$

be differentiable at a point $(x, y, z) \in G$, *and let the functions*

$$x = \varphi(t), \quad y = \psi(t), \quad z = \chi(t), \tag{2}$$

dependent on the scalar parameter t *have derivatives with respect to* t. *Then the derivative with respect to* t *of the composite function* $u = F(t) = f(\varphi(t), \psi(t), \chi(t))$ *(that is the derivative of* f *along the curve specified by equation* (2)) *is computed by the formula*

$$F'(t) = f'_x(\varphi(t), \psi(t), \chi(t))\,\varphi'(t) + f'_y(\varphi(t), \psi(t), \chi(t))\,\psi'(t) + f'_z(\varphi(t), \psi(t), \chi(t))\,\chi'(t),$$

or in a brief form

$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}. \tag{3}$$

Indeed, by the differentiability of f at (x, y, z), we have

$$\Delta u = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x, y, z) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z + o(\rho)$$
$$(\rho = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} \to 0), \tag{4}$$

for any sufficiently small increment $(\Delta x, \Delta y, \Delta z)$. Let us assign an increment Δt to the value t of the parameter to which, in accordance with equalities (2), there corresponds the point (x, y, z). This results in some increments Δx, Δy and Δz of functions (2). If these very increments are substituted into (4), then the left-hand member

of (4) becomes equal to the increment $F(t+\Delta t) - F(t) = \Delta u$ of the function F at the point t. On dividing (4) by Δt and passing to the limit, we obtain

$$F'(t) = \lim_{\Delta t \to 0} \frac{\Delta u}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left(\frac{\partial f}{\partial x} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \frac{\Delta y}{\Delta t} + \frac{\partial f}{\partial z} \frac{\Delta z}{\Delta t} + \frac{o(\rho)}{\Delta t} \right)$$

$$= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt},$$

i.e. (3), since functions (2) have derivatives and

$$\frac{o(\rho)}{\Delta t} = \varepsilon(\rho) \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}$$

$$\to 0 \cdot \sqrt{x_t'^2 + y_t'^2 + z_t'^2} = 0 \quad (\Delta t \to 0)$$

($\Delta t \to 0$ implies $\rho \to 0$).

Remark. If the functions x, y, z depend on several variables, say, of two:

$$x = \varphi(t, \tau), \quad y = \psi(t, \tau), \quad z = \chi(t, \tau),$$

then, fixing first τ and then t, on the basis of (3), we get

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t}, \quad \frac{\partial u}{\partial \tau} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \tau} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \tau} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \tau}.$$

Theorem 2. *If a function f is differentiable at a point (x, y, z), then its derivative in the direction of any unit vector $\mathbf{n} = (\cos\alpha, \cos\beta, \cos\gamma)$ exists and is expressed by the formula*

$$\frac{\partial f}{\partial \mathbf{n}} = \frac{\partial f}{\partial x} \cos\alpha + \frac{\partial f}{\partial y} \cos\beta + \frac{\partial f}{\partial z} \cos\gamma \qquad (5)$$

(α, β, γ are the angles formed by the vector $\mathbf{n}$ *with the x-, y-,* and *z-axes, respectively).*

Proof. According to the definition of the directional derivative (see Sec. 8.5) and by virtue of the foregoing

theorem, we have

$$\frac{\partial f}{\partial \mathbf{n}} = \lim_{\substack{t \to 0 \\ t > 0}} \frac{f(x+t\cos\alpha,\ y+t\cos\beta,\ z+t\cos\gamma) - f(x,\ y,\ z)}{t}$$

$$= \left[\frac{d}{dt} f(x+t\cos\alpha,\ y+t\cos\beta,\ z+t\cos\gamma)\right]_{t=0}$$

$$= \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta + \frac{\partial f}{\partial z}\cos\gamma,$$

where the partial derivatives are taken at the point (x, y, z).

If $x = \varphi(s)$, $y = \psi(s)$, $z = \chi(s)$ are equations of a smooth curve Γ, where the parameter s is the arc length, then the derivatives

$$\frac{dx}{ds} = \varphi'(s), \quad \frac{dy}{ds} = \psi'(s), \quad \frac{dz}{ds} = \chi'(s)$$

are equal to the direction cosines of the tangent vector to Γ. Therefore the quantity

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\varphi'(s) + \frac{\partial f}{\partial y}\psi'(s) + \frac{\partial f}{\partial z}\chi'(s) = \frac{d}{ds} f(\varphi(s),\ \psi(s),\ \chi(s)),$$

where f is a differentiable function, is the directional derivative of f along the indicated tangent vector. In such a case we also say that $\frac{\partial f}{\partial s}$ is the derivative of f along Γ.

Let us introduce the vector

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z}\right). \tag{6}$$

It is called the *gradient* of the function f at the point (x, y, z).

Formula (5) means that the *derivative of f in the direction of unit vector* $\mathbf{n}$ *at the point (x, y, z) is equal to the projection of the gradient of f at this point on that direction*:

$$\frac{\partial f}{\partial \mathbf{n}} = (\operatorname{grad} f,\ \mathbf{n}) = \operatorname{grad}_n f. \tag{7}$$

For any vector $\mathbf{n}$ there holds the obvious inequality

$$\frac{\partial f}{\partial \mathbf{n}} \leqslant |\operatorname{grad} f|. \tag{8}$$

If grad $f = \mathbf{0}$ (usually this occurs only at some singular points), then $df/d\mathbf{n} = 0$ for any vector $\mathbf{n}$. If grad $f \neq \mathbf{0}$, i.e. if at least one of the partial derivatives of f is different from zero, then (8) is a strict inequality for all unit vectors $\mathbf{n}$ except for the single vector $\mathbf{n}_0 = (\cos\alpha_0, \cos\beta_0, \cos\gamma_0)$ going in the direction of grad f $\left(\frac{\partial f}{\partial \mathbf{n}_0} = |\operatorname{grad} f| > 0\right)$. Thus,

$$\cos\alpha_0 = \frac{\frac{\partial f}{\partial x}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}},$$

$$\cos\beta_0 = \frac{\frac{\partial f}{\partial y}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}},$$

$$\cos\gamma_0 = \frac{\frac{\partial f}{\partial z}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}}. \tag{9}$$

It follows from the aforesaid that the *gradient of the function f at a point (x, y, z) can be defined as a vector possessing the following two properties*:

(1) *its length is equal to the maximum possible value of the directional derivative $df/d\mathbf{n}$* at (x, y, z) (this maximum value exists for any function differentiable at (x, y, z) and is a nonnegative number);

(2) *if its length is different from zero, then it goes along the vector* $\mathbf{n}$ *in whose direction the derivative $\partial f/\partial \mathbf{n}$ attains the maximum value.*

Sec. 8.9. Differential of a Function. Differentials of Higher Orders

Let us consider a function

$$W = f(\mathbf{x}) = f(x_1, \ldots, x_n), \tag{1}$$

defined on an open set $G \subset R_n$. It can be represented in infinitely many ways in the form

$$W = \varphi(\mathbf{u}) = \varphi(u_1, \ldots, u_m), \tag{2}$$

where

$$u_j = \psi_j(\mathbf{x}) \quad (j = 1, \ldots, m;\ \mathbf{x} \in G). \tag{3}$$

We shall use the following terminology: the variable W will be spoken of as a function of the *independent vector variable* $\mathbf{x}$, and the same variable W will be referred to as a function of the *dependent vector variable* $\mathbf{u}$. The latter variable ($\mathbf{u}$) depends on the independent variable $\mathbf{x}$; to each vector $\mathbf{x}$ belonging to G there corresponds the vector $\mathbf{u} = (\psi_1(\mathbf{x}), \ldots, \psi_m(\mathbf{x}))$.

Thus, the vector variable $\mathbf{x}$ will play an exceptional role: in all the considerations below it will enter *only as an independent variable.*

Let the function f possess continuous partial derivatives of the first order at a point $\mathbf{x} \in G$. Then, as is known from Sec. 8.6, it is differentiable, i.e. its increment at this point can be represented in the form

$$\Delta W = \sum_{j=1}^{n} \frac{\partial W}{\partial x_j} \Delta x_j + o(\rho) \quad (\rho \to 0), \tag{4}$$

$$\rho = |\Delta x| = \left(\sum_{j=1}^{n} \Delta x_j^2\right)^{1/2},$$

and its differential

$$dW = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \Delta x_j. \tag{5}$$

For independent $x_1, \ldots, x_n$ we set

$$\Delta x_j = dx_j \quad (j = 1, \ldots, n) \tag{6}$$

and call these quantities not only the increments of the independent variables x_i, but also their *differentials.* The differentials dx_j will also be called *independent differentials* in order to stress that they are independent of $\mathbf{x} = (x_1, \ldots, x_n)$. The "independence" of the quantities dx_j is displayed formally in the differentiation process: when differentiating with respect to $x_1, \ldots, x_n$, we regard them as *constants* ($d(dx_j) = 0$).

By virtue of the convention expressed by (6), the differential of W can be written in the form

$$dW = \sum_{j=1}^{n} \frac{\partial W}{\partial x_j} dx_j. \tag{7}$$

It is clear that dW is a quantity which, generally speaking, depends both on $x_1, \ldots, x_n$ and $dx_1, \ldots, dx_n$.

The general properties expressed by the relations below apply to any functions u and v having continuous partial derivatives at $\mathbf{x}$:

$$d(u \pm v) = du \pm dv, \tag{8}$$

$$d(uv) = u\,dv + v\,du, \tag{9}$$

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2} \quad (v \neq 0). \tag{10}$$

In these formulas the partial derivatives of the functions in the parenthesis are continuous at the point $\mathbf{x}$.

As an example, let us prove the third of these equalities:

$$d\left(\frac{u}{v}\right) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{u}{v}\right) dx_j = \sum_{j=1}^{n} \frac{v\dfrac{\partial u}{\partial x_j} - u\dfrac{\partial v}{\partial x_j}}{v^2} dx_j$$

$$= \frac{1}{v^2}\left(v\sum_{j=1}^{n} \frac{\partial u}{\partial x_j} dx_j - u\sum_{j=1}^{n} \frac{\partial v}{\partial x_j} dx_j\right) = \frac{v\,du - u\,dv}{v^2}.$$

The continuity of $\frac{\partial}{\partial x_j}\left(\frac{u}{v}\right)$ is seen from the third term in this chain of equalities.

The differential of W is also called the *differential of the first order* (below we also consider higher-order differentials).

Now let us suppose that the function W has continuous partial derivatives of the second order. The *second differential* of W corresponding to the independent increments (differentials) $dx_1, \ldots, dx_n$ is defined by the equality

$$d^2W = d(dW), \tag{11}$$

where the operation d used twice in the right-hand side of (11) is performed for the given independent increments $dx_1, \ldots, dx_n$ which are regarded as constants (independent of $x_1, \ldots, x_n$). Thus

$$d^2W = d\sum_{j=1}^{n} \frac{\partial W}{\partial x_j} dx_j = \sum_{j=1}^{n} d\left(\frac{\partial W}{\partial x_j} dx_j\right)$$

$$= \sum_{j=1}^{n} \left(d\frac{\partial W}{\partial x_j}\right) dx_j = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 W}{\partial x_i \, \partial x_j} dx_i \, dx_j. \qquad (12)$$

Since $\frac{\partial^2 W}{\partial x_i \, \partial x_j} = \frac{\partial^2 W}{\partial x_j \, \partial x_i}$, the second differential is a quadratic form in the independent differentials $dx_1, \ldots, dx_n$ which is a function of the form $\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ik}\xi_i\xi_k$, where $a_{ik} = a_{ki}$.

In general, the differential of the lth order of W taken for the independent differentials $dx_1, \ldots, dx_n$ is defined by induction with the aid of the recurrence formula

$$d^lW = d\,(d^{l-1}W) \quad (l = 2, 3, \ldots), \qquad (13)$$

where d^l, d, and d^{l-1} are taken for the indicated independent differentials dx_i, the latter being regarded as constants (independent of $x_1, \ldots, x_n$) in the computation process.

Reasoning as in (12), we easily obtain that

$$d^3W = \sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^3 W}{\partial x_k \, \partial x_i \, \partial x_j} dx_k \, dx_i \, dx_j.$$

Since we assume that the function f has continuous partial derivatives, the notation for the differentials can be simplified.

For instance, for the function of two variables $u = f(x, y)$ we have

$$d^2u = f''_{x^2}\, dx^2 + 2f''_{xy}\, dx\, dy + f''_{y^2}\, dy^2,$$

$$d^3u = d\,(d^2u) = \frac{\partial^3 f}{\partial x^3} dx^3 + 3\frac{\partial^3 f}{\partial x^2 \, \partial y} dx^2\, dy + 3\frac{\partial^3 f}{\partial x \, \partial y^2} dx\, dy^2 + \frac{\partial^3 f}{\partial y^3} dy^3.$$

Using the method of mathematical induction, we easily obtain that

$$d^n u = \frac{\partial^n f}{\partial x^n} dx^n + n \frac{\partial^n f}{\partial x^{n-1} \partial y} dx^{n-1} dy$$
$$+ \dots + \frac{n(n-1)\dots(n-k+1)}{k!} \frac{\partial^n f}{\partial x^{n-k} \partial y^k}$$
$$\times dx^{n-k} dy^k + \dots + \frac{\partial^n f}{\partial y^n} dy^n.$$

This can be written, using the following notation:

$$d^n u = d^n f = \left\{ \frac{\partial}{\partial x} dx + \frac{\partial}{\partial x} dy \right\}^n f,$$

where, in the right-hand member, we first raise the expression in braces to the nth power and then add f to the symbol ∂^n.

In the many-dimensional case there takes place a similar symbolic formula:

$$d^m f = \left\{ \sum_{i=1}^{n} \frac{\partial}{\partial x_i} dx_i \right\}^m f. \tag{14}$$

We have defined the notion of the differential of the function W in terms of the independent variables $x_1, \dots, x_n$ (or, which is the same, in terms of the vector variable $\mathbf{x}$). Now let us suppose that the variable W is regarded as a function of the *dependent* vector variable $\mathbf{u} = (u_1, \dots, u_m)$ (see the beginning of the present section). There arises the question concerning the expression of the first and higher differentials in terms of the variable $\mathbf{u}$. We begin the study of this question with the case of the first differential.

We shall assume that the functions $\varphi(\mathbf{u})$ and $\psi_j(\mathbf{x})$ $(j = 1, \dots, m)$ mentioned at the beginning of the section have continuous partial derivatives of the first order. Then

$$dW = \sum_{j=1}^{n} \frac{\partial W}{\partial x_j} dx_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \frac{\partial W}{\partial u_i} \frac{\partial u_i}{\partial x_j} \right) dx_j$$
$$= \sum_{i=1}^{m} \frac{\partial W}{\partial u_i} \sum_{j=1}^{n} \frac{\partial u_i}{\partial x_j} dx_j = \sum_{i=1}^{m} \frac{\partial W}{\partial u_i} du_i, \tag{15}$$

and we have obtained that, as in the case of one independent variable, the first differential of W is expressed in terms of the dependent variables in the same way as in terms of the independent variables. This demonstrates the *invariance of the form of the first differential.*

To answer the above stated question for the case of the second differential, we shall suppose that the functions φ and ψ_j possess continuous partial derivatives of the second order.

Differentiating both members of (15) and taking into consideration properties (8) and (9), we obtain (see the explanations below)

$$
\begin{aligned}
d^2W = d(dW) &= \sum_{i=1}^{m} d\left(\frac{\partial W}{\partial u_i}\, du_i\right) \\
&= \sum_{i=1}^{m}\left(\sum_{j=1}^{m} \frac{\partial^2 W}{\partial u_j\, \partial u_i}\, du_j\, du_i + \frac{\partial W}{\partial u_i}\, d^2u_i\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\partial^2 W}{\partial u_j\, \partial u_i}\, du_j\, du_i + \sum_{i=1}^{m} \frac{\partial W}{\partial u_i}\, d^2u_i. \qquad (16)
\end{aligned}
$$

In the second equality of this chain we have taken advantage of property (8), while in the third equality we have used property (9) and also the fact that the form of the first differential remains unchanged when we pass to the dependent variables u_j. We see that, in contrast to the first differential, the second differential of W expressed in terms of the dependent variables u_j is the sum of two summands of essentially different nature. The first summand is a quadratic form similar to form (12) expressing d^2W in terms of the independent variables. As to the second summand, it is an additional expression which should be taken into account: if u_j ($j = 1, \ldots, m$) is not a linear function of x_j, then this additional expression is not at all equal to zero.

Note that, as follows from our reasoning, if expression (16) is taken for the differentials $dx_1, \ldots, dx_n$, which enter into expression (12), then both expressions coin-

cide identically for any point $\mathbf{x}$ and any dx_i, provided that the indicated continuous partial derivatives of the second order exist at $\mathbf{x}$.

The expressions of the differentials d^3W, d^4W, ... in terms of the dependent variables u_j are found in a similar way, but they become more and more complicated as the order of the differential increases.

Sec. 8.10. Taylor's Formula

Let us confine ourselves to considering a function of two independent variables. Let $u = f(x, y)$ have in a neighbourhood of the point $P_0 = (x_0, y_0)$ continuous derivatives of all orders up to lth inclusive. Let us take the point $P_1 = (x_0 + \Delta x, y_0 + \Delta y)$ in this neighbourhood and join the points P_0 and P_1 by means of a line segment whose equation can be written in the parametric form as follows:

$$x = x_0 + t\,\Delta x, \quad y = y_0 + t\,\Delta y \quad (0 \leqslant t \leqslant 1).$$

Then, along this line segment, our function $u = f(x, y)$ will be a function of one variable t:

$$f(x, y) = f[x_0 + t\,\Delta x, y_0 + t\,\Delta y] = F(t). \tag{1}$$

It is easily seen that the difference

$$\Delta f(P_0) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = F(1) - F(0). \tag{2}$$

Maclaurin's formula for the function $F(t)$ in a neighbourhood of the point $t_0 = 0$ has the form

$$F(t) = F(0) + \frac{F'(0)}{1!}t + \ldots + \frac{F^{(l-1)}(0)}{(l-1)!}t^{l-1} + \frac{F^{l}(\theta)}{l!}t^{l}$$
$$(0 < \theta < t)$$

Putting $t = 1$, we obtain

$$F(1) - F(0) = \sum_{k=1}^{l-1} \frac{F^{(k)}(0)}{k!} + \frac{F^{(l)}(\theta)}{l!}, \quad \text{where} \quad 0 < \theta < 1. \tag{3}$$

Let us compute the derivatives of the function $F(t)$ in terms of $f(x, y)$. From relation (1) we have

$$F'(t) = \frac{\partial f(x_0 + t\,\Delta x,\ y_0 + t\,\Delta y)}{\partial x}\Delta x + \frac{\partial f(x_0 + t\,\Delta x,\ y_0 + t\,\Delta y)}{dy}\Delta y,$$

whence for $t = 0$ we obtain

$$F'(0) = \frac{\partial f(x_0, y_0)}{dx}\Delta x + \frac{\partial f(x_0, y_0)}{\partial y}\Delta y = df(P_0).$$

Quite analogously,

$$F''(t) = f''_{x^2}(x_0 + t\,\Delta x,\ y_0 + t\,\Delta y)\,\Delta x^2 + 2f''_{xy}(x_0 + t\,\Delta x,\ y_0 + t\,\Delta y)\,\Delta x\,\Delta y + f''_{y^2}(x_0 + t\,\Delta x,\ y_0 + t\,\Delta y)\,\Delta y^2, \quad F''(0) = d^2 f(P_0).$$

Proceeding in the same manner, we get

$$F'''(0) = d^3 f(P_0),\ \ldots,\ F^{(l-1)}(0) = d^{l-1} f(P_0).$$

By virtue of this, from (2) and (3) we have

$$\Delta f(P_0) = \frac{df(P_0)}{1!} + \ldots + \frac{d^{l-1} f(P_0)}{(l-1)!} + \frac{1}{l!} d^l f(x_0 + \theta\,\Delta x, y_0 + \theta\,\Delta y). \quad (4)$$

Formula (4) is called *Taylor's formula* for the function $u = f(x, y)$. It looks like the one for a function of one variable, but when expanded it is much more complicated.

For a function of n variables ($n > 2$) Taylor's formula is written in the same form (4).

For $l = 1$ Taylor's formula for a function f of n independent variables has the form

$$f(x) - f(x^0) = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}\right)_{\mathbf{x}^0 + \theta(\mathbf{x} - \mathbf{x}^0)} (x_j - x_j^0) \quad (0 < \theta < 1),$$

where the symbol $(\ )_{\mathbf{a}}$ means that the function in the parentheses is evaluated for the point $\mathbf{x} = \mathbf{a}$. This formula generalizes Lagrange's one-dimensional formula of finite increments to the case of dimension n.

For $l = 2$ and $n = 2$ the expanded form of formula (4) is written as follows:

$$f(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$
$$+ \frac{1}{2}\Big[\frac{\partial^2 f}{\partial x^2}(x_0 + \theta(x - x_0), y_0 + \theta(y - y_0))(x - x_0)^2$$
$$+ 2\frac{\partial^2 f}{\partial x\, \partial y}(x_0 + \theta(x - x_0), y_0 + \theta(y - y_0))(x - x_0)(y - y_0)$$
$$+ \frac{\partial^2 f}{\partial y^2}(x_0 + \theta(x - x_0), y_0 + \theta(y - y_0))(y - y_0)^2\Big].$$

For $l = 2$ and an arbitrary n formula (4) is written in the following way:

$$f(\mathbf{x}) = f(\mathbf{x}^0) + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)_{\mathbf{x}^0}(x_i - x_i^0)$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial^2 f}{\partial x_i\, \partial x_j}\right)_{\mathbf{x}^0 + \theta(\mathbf{x} - \mathbf{x}^0)}(x_i - x_i^0)(x_j - x_j^0),$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{x}^0 = (x_1^0, \ldots, x_n^0)$.

Sec. 8.11. Closed Sets

A set $A \subset R_n = R$ is said to be *bounded* if there exists a number $M > 0$ such that

$$|\mathbf{x}| \leqslant M, \quad \forall\, \mathbf{x} \in A,$$

or, in other words, if there exists a ball in R, with centre at the origin, containing A.

A set A is said to be *closed* if from the fact that some sequence of points $\mathbf{x}^k$ ($k = 1, 2, \ldots$), belonging to A, converges to the point $\mathbf{x}^0 \in R$ ($\mathbf{x}^k \to \mathbf{x}^0$, $\mathbf{x}^k \in A$) it follows that $\mathbf{x}^0$ belongs to A ($\mathbf{x}^0 \in A$).

It should be stressed that this definition does not require that A should contain a convergent sequence. It only requires that if A contains a convergent sequence, then the point to which it converges belongs to A.

This shows that it should be supposed that an empty set is closed. The entire space R_n is also, obviously, closed but unbounded.

As an example, let us consider an ellipsoid in the three-dimensional space

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad (a, b, c>0), \tag{1}$$

i.e. the set of points (x, y, z) satisfying equation (1). Let us denote this set by B. This is a bounded set, since for any of its points (x, y, z) the following inequality is fulfilled

$$x^2+y^2+z^2 \leqslant m\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=m \cdot 1=m,$$

where $m \geqslant a^2, b^2, c^2$. It is also closed, since, if we take an arbitrary sequence of points $(x_k, y_k, z_k) \in B$ tending to the point (x_0, y_0, z_0), then this latter also belongs to B. In fact, from the equality

$$\frac{x_k^2}{a^2}+\frac{y_k^2}{b^2}+\frac{z_k^2}{c^2}=1 \quad (k=1, 2, \ldots)$$

on passing to the limit as $k \to \infty$, there follows the equality

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}=1,$$

showing that $(x_0, y_0, z_0) \in B$.

Let us now consider a more extensive set A consisting of points (x, y, z) whose coordinates satisfy the inequality

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leqslant 1, \tag{2}$$

The set A is also, obviously, bounded. It is also closed, since if

$$(x_k, y_k, z_k) \in A \quad (k=1, 2, \ldots),$$

i.e.

$$\frac{x_k^2}{a^2}+\frac{y_k^2}{b^2}+\frac{z_k^2}{c^2} \leqslant 1$$

and $(x_k, y_k, z_k) \to (x_0, y_0, z_0)$, then, obviously,

$$\lim_{k \to \infty}\left(\frac{x_k^2}{a^2}+\frac{y_k^2}{b^2}+\frac{z_k^2}{c^2}\right)=\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2} \leqslant 1,$$

i.e. $(x_0, y_0, z_0) \in A$.

In connection with this, it is of interest to consider a third example: the set A' of points (x, y, z) with coordinates satisfying the strict inequality

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1. \tag{3}$$

The set A' is open (see Sec. 8.2), it is not closed. Let us take, for instance, a sequence of points $(\alpha_k, 0, 0)$, where α_k tends to the number a, strictly increasing. Then $(\alpha_k, 0, 0) \in A'$ $(k = 1, 2, \ldots)$ and $(\alpha_k, 0, 0) \to \to (a, 0, 0)$. But the limit point $(a, 0, 0)$ does not belong to A'.

The above considered examples are readily generalized. Let there be given a continuous functon $F(\mathbf{x}) = F(x_1, \ldots, x_n)$ in the entire space R_n. Then the set B of all points $\mathbf{x} = (x_1, \ldots, x_n)$, for which the inequality

$$F(\mathbf{x}) = F(x_1, \ldots, x_n) = C, \tag{4}$$

is fulfilled (C is an arbitrary number), is closed.

Indeed, it may happen that there are no points $\mathbf{x}$ at all, satisfying equality (4), i.e. B is an empty set, but, as we know, an empty set is closed. Let now B be a nonempty set and a certain sequence of points $\{\mathbf{x}^k\}$ belonging to B converge to the point $\mathbf{x}^0 \in R_n$ (if B consists even of one point $\mathbf{x}^0$, we may then construct a convergent sequence of points belonging to B, viz. $\{\mathbf{x}^0, \mathbf{x}^0, \ldots, \}$). Then $F(\mathbf{x}^k) = C$ $(k = 1, 2, \ldots)$, and, since F is continuous at the point $\mathbf{x}^0$,

$$\lim_{k\to\infty} F(\mathbf{x}^k) = F(\mathbf{x}^0) = C.$$

But then $\mathbf{x}^0 \in B$, i.e. set B is closed.

Similarly, the set of all points $\mathbf{x}$ satisfying the inequality $F(\mathbf{x}) \leqslant C$, where C is an arbitrary number and F is a function continuous in R_n, is closed, because from the relationships

$$F(\mathbf{x}^k) \leqslant C \quad (k = 1, 2, \ldots), \quad \mathbf{x}^k \to \mathbf{x}^0$$

(due to the continuity of F in R_n) it follows: $F(\mathbf{x}^0) \leqslant C$.

24—01779

By virtue of the aforesaid, the n-dimensional ellipsoid

$$\sum_{k=1}^{n} \frac{x_k^2}{a_k^2} = 1 \ (a_k > 0) \tag{5}$$

is a closed set in R_n.

The n-dimensional solid ellipsoid

$$\sum_{k=1}^{n} \frac{x_k^2}{a_k^2} \leqslant 1 \tag{6}$$

is also a closed set in R_n. But the set

$$\sum_{k=1}^{n} \frac{x_k^2}{a_k^2} < 1, \tag{7}$$

which can be naturally called an n-dimensional open solid ellipsoid, is not closed. We can get convinced by reasoning as in the case of formula (3). This set is an open set (see Sec. 8.2).

Let A be an arbitrary set belonging to R_n and let $\mathbf{x}^0$ be an arbitrary point of R_n ($A \subset R_n$, $\mathbf{x}^0 \in R_n$). Then, only three mutually exclusive cases are possible:

1. There exists an open sphere (ball) $V_{\mathbf{x}^0}$ with centre at the point $\mathbf{x}^0$ belonging to A ($V_{\mathbf{x}^0} \subset A$). In this case $\mathbf{x}^0$ is called an *interior point of the set A* (see Sec. 8.2).

2. There exists a sphere $V_{\mathbf{x}^0}$ with centre at $\mathbf{x}^0$ all points of which do not belong to A ($V_{\mathbf{x}^0} \subset R_n \setminus A$). In this case $\mathbf{x}^0$ is called an *exterior point of the set A*.

3. In any sphere $V_{\mathbf{x}^0}$ with centre at $\mathbf{x}^0$ there are points belonging to A and not belonging to A. In this case $\mathbf{x}^0$ is called a *boundary point of the given set A*.

The set A' of all the interior points of A is called the *interior* of A. This is an open set (see Sec. 8.2). If A' is not empty, then every point belonging to A' can be covered by a sphere with centre at this point, entirely belonging to A. If A' is an empty set, then it is formally regarded as open.

The set $\Gamma \equiv \partial A$ of all boundary points of A is called the *boundary of the set A*. This is a closed set since if $\mathbf{x}^k \to \mathbf{x}^0$ and $\mathbf{x}^k \in \Gamma$ ($k = 1, 2, \ldots$), then any open sphere

$V_{\mathbf{x}^0}$ with centre at $\mathbf{x}^0$ contains a certain point $\mathbf{x}^k$. The latter can be covered with a sphere $V_{\mathbf{x}^k}$, with centre at this point entirely belonging to $V_{\mathbf{x}^0}$ ($V_{\mathbf{x}^k} \subset V_{\mathbf{x}^0}$). But in $V_{\mathbf{x}^k}$ there are points belonging to A and not belonging to A, but then in $V_{\mathbf{x}^0}$ there are also points belonging to A and points not belonging to A. Consequently, $\mathbf{x}^0 \in \Gamma$.

The set A'' of all exterior points of the set A is, obviously. open.

The boundary points of A may belong and may not belong to the set A.

Shown in Fig. 99, the set $A \subset R_2$ consists of points (x_1, x_2):

$$x_1^2 + x_2^2 \leqslant 1, \; x_1 \leqslant 0,$$
$$x_1^2 + x_2^2 < 1, \; x_1 > 0,$$
$$x_1 = x_2 = 1.$$

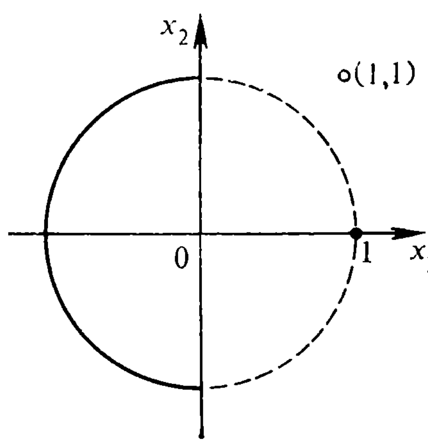

Fig. 99

It is evident that A' is the interior of a sphere of radius 1 with centre at the origin; Γ consists of the points of the circle $x_1^2 + x_2^2 = 1$ and the point (1, 1); A'' contains all the points lying outside the circle of unit radius, except the point (1, 1). Here the right-hand half of the circle does not belong to A but is a portion of the boundary Γ. Note that the given set A is neither open, nor closed.

Hence, if we are given an arbitrary set $A \subset R_n$, then, relative to this set, the space R_n can be represented in the form of a sum of the above defined three pairwise disjoint (nonintersecting) sets:

$$R_n = A' + \Gamma + A''.$$

If n-dimensional closed solid ellipsoid (6) is regarded as the set A, then A' is an open solid ellipsoid (7), and Γ is ellipsoid (5).

If A is an open set, then $R_n \setminus A$ is a closed set and vice versa. Indeed, let A be open and let $\mathbf{x}^k \to \mathbf{x}^0$, $\mathbf{x}^k \in R_n \setminus A$. If the point $\mathbf{x}^0$

belonged to A, then, by virtue of the fact that A is an open set, we could find a sphere $V_{\mathbf{x}^0}$ (with centre at $\mathbf{x}^0$) entirely belonging to A. But this is impossible, since there are points $\mathbf{x}^k$ in $V_{\mathbf{x}^0}$ which belong to $R_n \setminus A$. Hence, $\mathbf{x}^0 \in R_n \setminus A$ and $R_n \setminus A$ is closed.

Let now A be closed and let $\mathbf{x}^0 \in R_n \setminus A$. If the point $\mathbf{x}^0$ was a boundary point of A, then in any sphere $V_{\mathbf{x}^0}$ with centre $\mathbf{x}^0$ there would be points belonging to A. Then we could construct a sequence of points $\mathbf{x}^k \in A$ convergent to $\mathbf{x}^0$. But then, in consequence of the fact that A is closed, the point $\mathbf{x}^0$ would belong to A which contradicts the assumption that $\mathbf{x}^0 \in R_n \setminus A$. We have thus proved that an arbitrary point $\mathbf{x}^0 \in R_n \setminus A$ is an interior point of $R_n \setminus A$, i.e. that $R_n \setminus A$ is an open set.

The set $A + \Gamma$ is called the *closure* of A and is denoted as follows:

$$\overline{A} = A + \Gamma.$$

Obviously,

$$A' + \Gamma = A + \Gamma$$

since, on the one hand, $A' \subset A$ and, consequently, $A' + \Gamma \subset \subset A + \Gamma$, and, on the other hand, if $\mathbf{x} \in A + \Gamma$, then either $\mathbf{x} \in \Gamma$, and $\mathbf{x} \in A' + \Gamma$, or $\mathbf{x} \in A$ and $\mathbf{x} \bar{\in} \Gamma$, but then $\mathbf{x} \in A' \subset \subset A' + \Gamma$.

Further, $\overline{A} = A + \Gamma$ is a closed set, since the exterior of $A + \Gamma = A' + \Gamma$ is an open set.

Thus, in order to obtain $\overline{A}$ it is necessary to add to A all its boundary points not belonging to the set A.

If A is closed, then

$$A = A + \Gamma = \overline{A},$$

i.e. all the boundary points of A belong to A. In fact, $R_n \setminus A$ is open and every point $\mathbf{x}^0 \in R_n \setminus A$ can be covered with a sphere $V_{\mathbf{x}^0}$ containing no points belonging to A. And conversely, if

$$A = A + \Gamma = \overline{A},$$

then A is closed, because if $\mathbf{x}^k \to \mathbf{x}^0$, $\mathbf{x}^k \in A$, and if we assume that $\mathbf{x}^0 \bar{\in} A$, then we shall arrive at a contradiction, since then $\mathbf{x}^0 \in \Gamma \subset A + \Gamma = A$.

Thus *for a set A to be closed, it is necessary and sufficient that its closure coincide with it* ($A = \overline{A}$).

In particular, $\overline{A}$ is always closed, and therefore $\overline{\overline{A}} = \overline{A}$.

Finally, it should be noted that an empty set and the entire space R_n are simultaneously open and closed. We can prove that in all other cases if A is open, then it is not closed, and if A is closed, then it is not open.

Sec. 8.12. Functions Continuous on Bounded Closed Sets

Let A be, for the time being, an arbitrary set in the space $R = R_n$, and let a function $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ be defined on A. By definition, a *function f is continuous on A at a point* $\mathbf{x}^0 \in A$ *if*

$$\lim_{\substack{\mathbf{x}^k \to \mathbf{x}^0 \\ \mathbf{x}^k \in A}} f(\mathbf{x}^k) = f(\mathbf{x}^0), \tag{1}$$

for any sequence of points $\mathbf{x}^k \in A$ *convergent to* $\mathbf{x}^0$.

This definition of the continuity of a function at a point $\mathbf{x}^0$ differs from the ordinary definition given in Sec. 8.4 by that in the ordinary definition it was supposed that the function f is defined in a certain neighbourhood of the point $\mathbf{x}^0$ and it was required that limit (1) exist and be equal to $f(\mathbf{x}^0)$ for any sequence of points $\mathbf{x}^k$ *convergent* to $\mathbf{x}^0$.

Now we do not require that a function f be defined in the entire neighbourhood of $\mathbf{x}^0$. It is only required that f be defined at a point $\mathbf{x}^0 \in A$ and that limit (1) exist for any sequence of points $\mathbf{x}^k \in A$ convergent to the point $\mathbf{x}^0$.

The above given definition can also be formulated in terms of ε, δ: a function f is continuous at a point $\mathbf{x}^0 \in A$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(\mathbf{x}) - f(\mathbf{x}^0)| < \varepsilon, \qquad \forall \mathbf{x} \in A, \qquad |\mathbf{x} - \mathbf{x}^0| < \delta.$$

Now we shall assume that A is a bounded closed set in the space R and a given on A function $f(\mathbf{x})$ is continuous on this set. Under these assumptions we can prove the following remarkable properties:

(1) A function f is bounded on the set A.

(2) A function f attains its maximum and minimum on the set A, i.e there exist in A points $\mathbf{x}^0$ and $\mathbf{y}^0$ such that

$$f(\mathbf{x}^0) = \max_{\mathbf{x} \in A} f(\mathbf{x}), \quad f(\mathbf{y}^0) = \min_{\mathbf{x} \in A} f(\mathbf{x}).$$

(3) A function f is uniformly continuous on the set A, i.e. for any given $\varepsilon > 0$ there can be found $\delta > 0$ such that

$$|f(\mathbf{x}') - f(\mathbf{x}'')| < \varepsilon$$

for any $\mathbf{x}'$, $\mathbf{x}'' \in A$, satisfying the inequalities $|\mathbf{x}' - \mathbf{x}''| < \delta$.

As we see, properties (1), (2) and (3) generalize the known properties of a continuous function $f(x)$ of one variable x defined on a closed interval $[a, b]$. We should like to underline that the interval $[a, b]$ is a bounded closed one-dimensional set. In fact, if a sequence of points (numbers) x_k belonging to the closed interval $[a, b]$ converges to some point (number) x_0, then this point belongs to $[a, b]$ $(x_0 \in [a, b])$.

The proof of properties (1), (2), (3) is quite analogous to their proof for a closed interval $[a, b]$ discussed in Secs. 3.5 and 3.7. It is entirely based on the following lemma which generalizes the one-dimensional Bolzano-Weierstrass theorem from Sec. 2.9.

Lemma. *Any bounded sequence of points* $\mathbf{x}^k = (x_1^k, \ldots, x_n^k)$ $(k = 1, 2, \ldots)$ *contains a subsequence* $\{\mathbf{x}^{k_l}\}$ $(l = 1, 2, \ldots)$, *convergent to some point* $\mathbf{x}^0$:

$$|\mathbf{x}^{k_l} - \mathbf{x}^0| \to 0 \quad (l \to \infty).$$

Proof. Since the sequence $\{\mathbf{x}^k\}$ is bounded, there exists a number M such that

$$M \geqslant |\mathbf{x}^k| \geqslant |x_j^k| \quad (j = 1, \ldots, n; \quad k = 1, 2, \ldots).$$

This shows that the coordinates of the points $\mathbf{x}^k$ are also bounded. The first coordinate forms a bounded sequence $\{x_1^k\}$ $(k = 1, 2, \ldots)$, and, according to the one-dimensional Bolzano-Weierstrass theorem, there can be found a subsequence k_{l_1} of natural numbers and a certain number x_1^0 such that $x_1^{k_{l_1}} \to x_1^0$ $(l_1 \to \infty)$. The second coordinate x_2^k will be considered only for the found natural k_{l_1}. The subsequence $\{x_2^{k_{l_1}}\}$ is bounded, therefore it can also contain a subsequence $\{x_2^{k_{l_2}}\}$ and a number x_2^0 such that $x_2^{k_{l_1}} \to x_2^0$. Since $\{k_{l_2}\}$ is a subsequence contained in $\{k_{l_1}\}$, we have simultaneously: $x_1^{k_{l_2}} \to x_1^0$, $x_2^{k_{l_2}} \to x_2^0$. Proceeding in the same way, at the nth stage we shall obtain a subsequence of natural numbers $k_{l_n} = k_l$ and a system of numbers $x_1^0, x_2^0, \ldots, x_n^0$, such that the following takes place simultaneously:

$$x_1^{k_l} \to x_1^0, \quad x_2^{k_l} \to x_2^0, \quad \ldots, \quad x_n^{k_l} \to x_n^0 \quad (l \to \infty).$$

Putting $\mathbf{x}^0 = (x_1^0, \ldots, x_n^0)$, we obtain the assertion of the lemma.

Sec. 8.13. Extrema

Let on a domain (an open connected set) G there be given a function $u = f(\mathbf{x})$, $\mathbf{x} = (x_1, \ldots, x_n)$ and let $\mathbf{x}^0 = (x_1^0, \ldots, x_n^0)$ be a point of G. We say that the function $u = f(\mathbf{x})$ has a *local maximum* (*minimum*) at the point $\mathbf{x}^0$ if there exists a neighbourhood of this point such that for any $\mathbf{x}$ from this neighbourhood there takes place the inequality

$$f(\mathbf{x}) \leqslant f(\mathbf{x}^0) \quad (f(\mathbf{x}) \geqslant f(\mathbf{x}^0)). \tag{1}$$

The point $\mathbf{x}^0$ will be called a *point of local maximum* (*minimum*) and a particular value $f(\mathbf{x}^0)$ of the function will be referred to as a *maximum* (*minimum*) *value of the function*. A local maximum and a local minimum are usually called a *local extremum* which is a general term for both of them. It follows from the definition of an extremum that in a sufficiently small neighbourhood of the point $\mathbf{x}^0$ the increment of the function $\Delta u = f(\mathbf{x}) - f(\mathbf{x}^0)$ does not change its sign:

$\Delta u \geqslant 0$ in the case of a local minimum (min),

$\Delta u \leqslant 0$ in the case of a local maximum (max).

Theorem 1 (a necessary condition for an extremum). *Let a function $u=f(\mathbf{x})$ have a local extremum at a point $\mathbf{x}^0$. Then, if there exist partial first-order derivatives $\frac{\partial f}{\partial x_i}$ $(i = 1, \ldots, n)$ at $\mathbf{x}^0$, all of them vanish at this point*:

$$\frac{\partial f(\mathbf{x}^0)}{\partial x_i} = 0 \quad (i = 1, \ldots, n). \tag{2}$$

Proof. Let us prove that $\frac{\partial f(\mathbf{x}^0)}{\partial x_1} = 0$. Fixing the variables $x_2 = x_2^0, \ldots, x_n = x_n^0$, we obtain a function $u = f(x_1^0, x_2^0, \ldots, x_n^0)$ of one variable x_1, this function having a local extremum at the point x_1^0. Therefore, by virtue of a necessary condition for an extremum for a function of one variable, we conclude that the derivative of this function with respect to the variable x_1 must be equal to zero at the point x_1^0. But this derivative is a partial derivative of the function $f(\mathbf{x})$ with respect to the variable x_1

at the point $\mathbf{x}^0$, i.e.

$$\frac{\partial f(x_1^0, x_2^0, \ldots, x_n^0)}{\partial x_1} = \frac{\partial f(\mathbf{x}^0)}{\partial x_1} = 0.$$

Other cases are considered in a similar manner.

Corollary. *If a function* $u = f(\mathbf{x})$ *has an extremum at a point* $\mathbf{x}^0$ *and is differentiable at that point, then* $df(\mathbf{x}^0) = 0$ *or* $\operatorname{grad} f(\mathbf{x}^0) = \mathbf{0}$.

This corollary follows from the definitions of the differential and gradient.

Remark. Condition (2) is not sufficient for an extremum of the function f to occur at the point $\mathbf{x}^0$.

For instance, the function $u = x^2y$ has partial derivatives $\frac{\partial u}{\partial x} = 2xy$ and $\frac{\partial u}{\partial y} = x^2$ which vanish at the point (0, 0). But the point (0, 0) is not a point of extremum, since in any neighbourhood of this point $\Delta u = x^2y - 0 = x^2y$ attains both positive and negative values.

Points at which there exist continuous partial derivatives of f, satisfying system (2), will be called *stationary points*.

Let us pass to obtaining sufficient conditions for an extremum. Let a function $u = f(\mathbf{x})$ have continuous derivatives up to the second order (inclusively) with respect to all the variables, and let $\mathbf{x}^0$ be a stationary point, i.e. $df(\mathbf{x}^0) = 0$. Then, expanding the function $u = f(\mathbf{x})$ by means of Taylor's formula in a neighbourhood of the point $\mathbf{x}^0$, we obtain

$$\Delta f(\mathbf{x}^0) = df(\mathbf{x}^0) + \frac{1}{2!} d^2 f(\mathbf{x}^0 + \theta\Delta\mathbf{x}) = \frac{1}{2} d^2 f(\mathbf{x}^0 + \theta\Delta\mathbf{x})$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} f''_{x_i x_j}(\mathbf{x}^0 + \theta\Delta\mathbf{x})\, \Delta x_i\, \Delta x_j$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (f''_{x_i x_j}(\mathbf{x}^0) + \varepsilon_{ij})\, \Delta x_i\, \Delta x_j$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} f''_{x_i x_j}(\mathbf{x}^0)\, \Delta x_i\, \Delta x_j + \frac{\rho^2}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \varepsilon_{ij} \frac{\Delta x_i}{\rho} \cdot \frac{\Delta x_j}{\rho}$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} f''_{x_i x_j}(\mathbf{x}^0)\, \Delta x_i\, \Delta x_j + \frac{\rho^2}{2} \alpha(\Delta\mathbf{x}),$$

where $0 < \theta < 1$, $\Delta x = (\Delta x_1 \ldots, \Delta x_n)$, $\rho = |\Delta x| = \sqrt{\sum_{i=1}^{n} \Delta x_i^2}$, $\alpha(\Delta \mathbf{x}) \to 0$ as $\rho \to 0$.

Since the second derivatives are continuous, the quantities ε_{ij}, dependent on $\Delta \mathbf{x}$, tend to zero as $\rho = |\Delta \mathbf{x}| \to 0$, and then also $\max\limits_{i,j} |\varepsilon_{ij}| = \varepsilon \to 0$ as $\rho \to 0$. Therefore, taking into account that $\frac{|\Delta x_i|}{\rho} \leqslant 1$, we obtain

$$|\alpha(\Delta \mathbf{x})| = \left|\sum_{i=1}^{n} \sum_{j=1}^{n} \varepsilon_{ij} \frac{\Delta x_i}{\rho} \frac{\Delta x_j}{\rho}\right| \leqslant \varepsilon \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} 1 \cdot 1 = n^2 \varepsilon \to 0.$$

$$(\rho \to 0).$$

And so, we have proved that

$$\Delta f(\mathbf{x}^0) = f(\mathbf{x}^0 + \xi) - f(\mathbf{x}^0)$$
$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \xi_i \xi_j + \frac{\rho^2}{2} \alpha(\xi), \qquad (3)$$

where

$$a_{ij} = a_{ji} = f''_{x_i x_j}(\mathbf{x}^0), \quad \xi_i = \Delta x_i, \quad \xi = (\xi_1, \ldots, \xi_n)$$

and

$$\alpha(\xi) \to 0 \text{ as } \rho = |\xi| = \sqrt{\sum_{i=1}^{n} \xi_i^2} \to 0.$$

The expression

$$A(\xi) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \xi_i \xi_j \quad (a_{ij} = a_{ji}) \qquad (4)$$

is a quadratic form with respect to $\xi = (\xi_1, \ldots, \xi_n)$. By the sign of this form, we can find out the sign of $\Delta f(\mathbf{x}^0)$ for sufficiently small $|\Delta \mathbf{x}|$ with the aid of formula (3).

The following assertions hold true:

(1) If the form $A(\xi)$ is strictly positive definite, i.e. $A(\xi) > 0$ for all $\xi \neq 0$, then the function f has a local minimum at the point $\mathbf{x}^0$.

(2) If the form $A(\xi)$ is strictly negative definite, i.e. $A(\xi) < 0$ for all $\xi \neq 0$, then the function f has a local maximum at the point $\mathbf{x}^0$.

(3) If $A(\xi) \geqslant 0$ for all ξ or $A(\xi) \leqslant 0$ for all ξ and there is $\xi \neq 0$, for which $A(\xi) = 0$ then the question about the local extremum of the function f at the point $\mathbf{x}^0$ remains unanswered.

(4) If the form $A(\xi)$ is not defined with respect to sign, i.e. there exist vectors ξ' and ξ'', for which $A(\xi') > 0$, $A(\xi'') < 0$, then the function f has no local extremum at the point $\mathbf{x}^0$.

The Proof of Assertion (1). Let us rewrite equality (3) as follows:

$$\Delta f(\mathbf{x}^0) = \frac{\rho^2}{2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\xi_i}{\rho}\frac{\xi_j}{\rho} + \alpha(\xi)\right]$$

$$= \frac{\rho^2}{2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\eta_i\eta_j + \alpha(\xi)\right] = \frac{\rho^2}{2}[A(\eta) + \alpha(\xi)], \qquad (5)$$

where we have introduced new variables

$$\eta_i = \xi_i/\rho \quad (i = 1, 2, \ldots, n).$$

It is easy to see that

$$|\eta| = \sqrt{\sum_{i=1}^{n} \eta_i^2} = \frac{\sqrt{\sum_{i=1}^{n} \xi_i^2}}{\rho} = 1.$$

Thus, for any ξ the point η is found on the surface of the n-dimensional unit sphere. The function $A(\eta)$ is continuous on this surface representing a bounded closed set, and, by the hypothesis, is positive on this surface. But then $A(\eta)$ attains its minimum m at some point of this surface which is greater than zero ($m > 0$) (see Sec. 8.12, Property (2)). Since $\alpha(\xi) \to 0$ as $\rho = |\xi| \to 0$, for a sufficiently small $\delta > 0$ we have

$$|\alpha(\xi)| < m, \qquad \forall \xi : |\xi| < \delta.$$

Consequently,

$$\Delta f(\mathbf{x}^0) = f(\mathbf{x}^0 + \boldsymbol{\xi}) - f(\mathbf{x}^0) = \frac{\rho^2}{2}[A(\boldsymbol{\eta}) + \alpha(\boldsymbol{\xi})]$$

$$\geqslant \frac{\rho^2}{2}[m + \alpha(\boldsymbol{\xi})] \geqslant 0, \quad \forall \boldsymbol{\xi}:\ |\boldsymbol{\xi}| < \delta$$

and the function f has a local minimum at the point $\mathbf{x}^0$.

Assertion (2) is proved in an analogous way.

The Proof of Assertion (3). In this case the form $A(\boldsymbol{\xi})$ vanishes for some $\boldsymbol{\xi}' \neq 0$; but then, by virtue of the homogeneity property of the form $(A(\alpha\boldsymbol{\xi}) = \alpha^2 A(\boldsymbol{\xi}))$ for $\boldsymbol{\xi} = \alpha\boldsymbol{\xi}'$ where α is a number, it must also be equal to zero. This shows that for all indicated points $\boldsymbol{\xi}$ our form equals zero and, consequently, $f(\mathbf{x}^0 + \boldsymbol{\xi}) - f(\mathbf{x}^0) = \rho^2\alpha(\boldsymbol{\xi})$. But the sign of $\alpha(\boldsymbol{\xi})$ is unknown, therefore we cannot say whether or not f has an extremum at $\mathbf{x}^0$.

The Proof of Assertion (4). Here it is also convenient to make use of equality (5). In this case, by the hypothesis, there exists a point $\boldsymbol{\xi}'$, for which the form is positive and there exists a point $\boldsymbol{\xi}''$, for which the form is negative, but then for the corresponding points $\boldsymbol{\eta}' = \boldsymbol{\xi}'/\rho$, $\boldsymbol{\eta}'' = \boldsymbol{\xi}''/\rho$ there will be fulfilled the inequalities $A(\boldsymbol{\eta}') > 0$, $A(\boldsymbol{\eta}'') < 0$ and for small ρ's it will turn out that $A(\boldsymbol{\eta}') + \alpha(\boldsymbol{\xi}') > 0$, $A(\boldsymbol{\eta}'') + \alpha(\boldsymbol{\xi}'') < 0$, i.e. in any small neighbourhood of $\mathbf{x}^0$ there are points $\mathbf{x}'$ and $\mathbf{x}''$ for which $f(\mathbf{x}') > f(\mathbf{x}^0)$ and $f(\mathbf{x}'') < f(\mathbf{x}^0)$, and this automatically means that there is no extremum.

Here we refer to the well-known Sylvester conditions* expressed in terms of coefficients a_{ij} for which quadratic form (4) satisfy the above listed conditions (1)-(4). Here we are going to mention only the criteria following from Sylvester's theorem for the case of a function $u = f(x_1, x_2)$ of two variables.

If $a_{11} = f''_{x_1^2}(\mathbf{x}^0) > 0$ and

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2 = f''_{x_1^2}(\mathbf{x}^0)\, f''_{x_2^2}(\mathbf{x}^0) - [f''_{x_1x_2}(\mathbf{x}^0)]^2 > 0$$

---

* See our book *Fundamentals of Linear Algebra and Analytical Geometry*, Sec. 21.

(in this case form (4) is strictly positive definite), then the function $u = f(x_1, x_2)$ has a local minimum at the point $\mathbf{x}^0 = (x_1^0, x_2^0)$.

If

$$f''_{x_1^2}(\mathbf{x}^0) < 0, \quad f'_{x_1^2}(\mathbf{x}^0)\, f''_{x_2^2}(x^0) - [f''_{x_1x_2}(\mathbf{x}^0)]^2 > 0$$

(in this case form (4) is strictly negative definite), then the function $u = f(x_1, x_2)$ has a local maximum at the point $\mathbf{x}^0$.

If $a_{11}a_{22} - a_{12}^2 < 0$, then $d^2f(\mathbf{x}^0)$, as a quadratic form, is not definite by sign as Δx_i varies, therefore in this case $\Delta f(\mathbf{x}^0)$ does not retain sign either in any neighbourhood of the point $\mathbf{x}^0$, and consequently, there is no extremum at the point $\mathbf{x}^0$.

If $a_{11}a_{22} - a_{12}^2 = 0$, then the question concerning extrema remains open.

Example 1. For the function $u = x^3 - 3x + y^2$ the points $(\pm 1, 0)$ are stationary. Let us investigate them for extremum. We have

$$u''_{x^2} = 6x, \quad u''_{x^2}(\pm 1, 0) = \pm 6, \quad u''_{xy} = 0, \quad u''_{y^2} = 2.$$

Hence, for the point (1, 0): $a_{11}a_{22} - a_{12}^2 = 6 \times 2 - 0 = 12 > 0$, $a_{11} = 6 > 0$. Therefore, at the point (1, 0) our function has a local minimum. For the point (—1, 0): $a_{11} = -6 < 0$, $a_{11}a_{22} - a_{12}^2 = -12 < 0$, therefore the function has no extremum at this point.

Example 2. For the function $u = x^4 + y^2$ the point (0, 0) is a stationary one, and it is easily seen that the function has a local minimum at this point. Meanwhile, $u''_{x^2} = 12x^2$, $u''_{xy} = 0$, $u''_{y^2} = 2$, i. e. $a_{11} = 0$, $a_{12} = 0$, $a_{22} = 2$, and $a_{11}a_{22} - a_{12}^2 = 0$.

Example 3. For the function $u = x^3 + y^2$ at the stationary point (0, 0) we have: $a_{11} = 0$, $a_{12} = 0$, $a_{22} = 2$, that is, once again $a_{11}a_{22} - a_{12}^2 = 0$. But in this case the function $u = x^3 + y^2$ has no extremum at the point (0, 0), since on the straight line $y = 0$ the increment $\Delta u = x^3$ changes sign when passing through the point $x = 0$.

Sec. 8.14. Finding the Greatest and the Least Values of Functions

Let there be given a continuously differentiable function $u = f(x)$ on a set $G \subset R_n$ representing a closure of a bounded domain, that is a domain to which the boundary ∂G is added. Then f attains a maximum and a minimum at some points $\mathbf{x} \in G$ (see Sec. 8.12, Property (3)). These points can be either interior or boundary. If $\mathbf{x}$ is an interior point, then the function $f(\mathbf{x})$ has a local extremum at this point. Therefore, in order to find the greatest (the least) value of a function, it is necessary to find all its stationary points, to compute the values of the function at these points and to compare them with the values of the function on the boundary ∂G. The greatest of these values will be just the *greatest value* of the function on G.

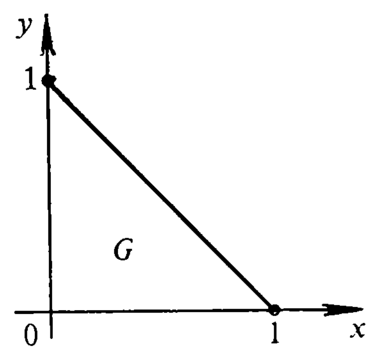

Fig. 100

If $G \subset R_2$ and ∂G is a continous plane curve $x = \varphi(t)$, $y = \varphi(t)$, then along this boundary our function is a function of one variable t: $f[\varphi(t), \psi(t)]$. We already know how to find the greatest value of this function.

Example. Find the greatest value of the function $z = 1 - x + x^2 + 2y$ in a closed domain G bounded by straight lines: $x = 0$, $y = 0$, and $x + y = 1$ (Fig. 100).

Solution. $z'_x = -1 + 2x = 0$, $z'_y = 2 \neq 0$, i.e. there are no stationary points. We now investigate the function z for ∂G.

(1) Let $x = 0$, then $z = 1 + 2y$, $0 \leqslant y \leqslant 1$. On $[0, 1]$ the function $z = 1 + 2y$ has no stationary points either, and $z(0) = 1$, $z(1) = 3$.

(2) Let $y = 0$, then $z = 1 - x + x^2$, $0 \leqslant x \leqslant 1$. Further, $z'_x = -1 + 2x = 0$ for $x = 1/2$, i.e. $x = 1/2$ is a stationary point. Evaluating the function at this point and at the boundary points $x = 0$ and $x = 1$, we obtain: $z(1/2) = 3/4$, $z(0) = 1$, $z(1) = 1$.

(3) Let $x + y = 1$, then $z = 3 - 3x + x^2$, $0 \leqslant x \leqslant 1$. Since $z'_x = -3 + 2x = 0$ at the point $x = 3/2 \overline{\in} [0, 1]$ there are no stationary points in the interval $[0, 1]$, Further, $z(0) = 3$, $z(1) = 1$.

Comparing all the greatest values of the function obtained for different portions of the boundary, we see that the greatest value of the function $z(x, y)$ on G is equal to 3 and is attained at the point $x^0 = (0, 1)$.

Sec. 8.15. Existence Theorem for an Implicit Function

Let us take an arbitrary function $f(x, y)$ of two variables x and y and equate it to zero:

$$f(x, y) = 0. \tag{1}$$

Let $\mathfrak{M}$ denote the set of all the points (x, y) for which equality (1) is fulfilled and let (x_0, y_0) be a point belonging to this set, i.e. $f(x_0, y_0) = 0$.

If no additional conditions are imposed on the function f, then the set $\mathfrak{M}$ can be of an arbitrary structure. For instance, in the case of the function $f(x, y) = (x - x_0)^2 + (y - y_0)^2$ the set $\mathfrak{M}$ consists of a single point (x_0, y_0); in the case of the function $f(x, y) = x^2 + y^2 + 1$ the set $\mathfrak{M}$ is empty; and if

$$f(x, y) = (x - x_0)^2 - (y - y_0)^2 = (x + y - x_0 - y_0) \times (x - y - x_0 + y_0),$$

then $\mathfrak{M}$ is a pair of straight lines passing through (x_0, y_0). We also frequently encounter the case when, at least in a sufficiently small neighbourhood of (x_0, y_0), the set $\mathfrak{M}$ represents a curve described by a continuous (one-valued) function

$$y = \psi(x), \ x \in (x_0 - \delta, x_0 + \delta)$$

(hence, ψ is a function represented implicitly by equation (1), see also Sec. 3.1).

In this connection there arises the question as to how we can determine which of the cases takes place judging by the properties of the function f. Proved below are two theorems giving answers to this question.

Theorem 1. *Let there be given an equation*

$$f(x, y) = 0, \tag{1}$$

satisfying the following properties.

The function f is defined in a two-dimensional neighbourhood Ω of the points (x_0, y_0) in the xy-plane and is continuous in that neighbourhood together with its partial derivatives of the first order, the derivative of f with respect to y is different from zero:

$$f'_y(x_0, y_0) = \left(\frac{\partial f}{\partial y}\right)_0 \neq 0 \tag{2}$$

and $f(x_0, y_0) = 0$. Let $\mathfrak{M}$ be the set of all points (x, y) satisfying equation (1) (in particular, $(x_0, y_0) \in \mathfrak{M}$).

Then for any $b_0 > 0$ there is a rectangle

$$\Delta = \{|x - x_0| < a, \ |y - y_0| < b\}, \ b < b_0, \tag{3}$$

belonging to Ω such that the set $\mathfrak{M}\Delta$ (the intersection of $\mathfrak{M}$ and Δ) is described by a continuously differentiable implicit function

$$y = \psi(x), \quad x \in \Delta^0, \tag{4}$$

$$\Delta^0 = \{|x - x_0| < a\}. \tag{5}$$

In other words, the rectangle Δ possesses the property that on its projection Δ^0 on the x-axis it is possible to define a continuously differentiable function (4) being the solution of equation (1), i.e. satisfying equation (1)

$$f(x, \psi(x)) \equiv 0, \quad x \in \Delta^0. \tag{6}$$

Its graph is entirely contained in Δ. This function is determined uniquely in the sense that the coordinates of any point $(x, y) \in \mathfrak{M}\Delta$ are related by equation (4). In particular, $y_0 = \psi(x_0)$, since $(x_0, y_0) \in \mathfrak{M}\Delta$ (Fig. 101),

Proof of Theorem 1. Let, for the sake of definiteness. $f'_y(x_0, y_0) > 0$. Since f'_y is continuous in Ω, there exists a neighbourhood of the point (x_0, y_0) which will be denoted again by Ω, such that in it $f'_y(x, y) > 0$. Let us introduce a closed rectangle

$$\bar{\Delta} = \{|x - x_0| \leqslant \bar{a}, \ |y - y_0| \leqslant b\} \subset \Omega \quad (b < b_0).$$

Then $f'_y(x, y) > 0$ in $\bar{\Delta}$ and

$$\min_{(x, y)\in\bar{\Delta}} f'_y(x, y) = m > 0. \tag{7}$$

The function $f(x, y)$ considered on the closed interval $[y_0 - b \leqslant y \leqslant y_0 + b,\ x = x_0]$, as a function of y is

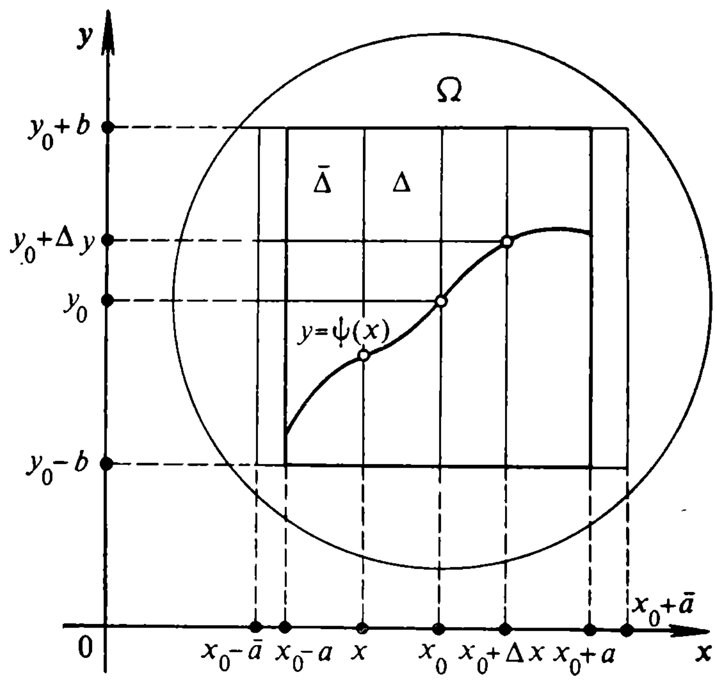

Fig. 101

continuous, strictly increasing and vanishes at the point $y = y_0$ (by the conditions of the theorem, $f(x_0, y_0) = 0$). Hence,

$$f(x_0, y_0 - b) < 0, \quad f(x_0, y_0 + b) > 0.$$

Since f is continuous, we can find a sufficiently small number a, $0 < a < \bar{a}$, such that

$$f(x, y_0 - b) < 0,\ f(x, y_0 + b) > 0, \quad \forall\, x \in \Delta^0 = \{|x - x_0| < a\}.$$

Let us denote the open rectangle by $\Delta = \{|x - x_0| < a,\ |y - y_0| < b\}$. Obviously, $\Delta \subset \bar{\Delta} \subset \Omega$ and Δ^0 is the projection of Δ on the x-axis.

Let us now choose an arbitrary point $x \in \Delta^0$, fix it temporarily and consider the function $f(x, y)$ as a function of y on the closed interval $[y_0 - b, y_0 + b]$. It is continuous, strictly increasing (since $f'_y > 0$) and assumes values of opposite signs at the end points of the interval. But then, by the theorem on an intermediate value, there exists a single value (number) y belonging to the interval $(y_0 - b, y_0 + b)$ which we denote $y = \psi(x)$ for which $f(x, \psi(x)) = 0$.

This proves the existence of a function $\psi(x)$ defined on Δ^0 and satisfying equation (6).

Let us prove that the function $\psi(x)$ is continuous on Δ^0. Let x, $x + \Delta x \in \Delta^0$, $y = \psi(x)$, $\Delta y = \psi(x + \Delta x) - \psi(x)$. Then, on the basis of Taylor's formula, we have (see Sec. 8.10)

$$0 = f(x+\Delta x, y+\Delta y) - f(x, y)$$
$$= f'_x(x+\theta\Delta x, y+\theta\Delta y)\Delta x + f'_y(x+\theta\Delta x, y+\theta\Delta y)\Delta y,$$

where $0 < \theta < 1$. Hence,

$$\frac{\Delta y}{\Delta x} = -\frac{f'_x(x+\theta\Delta x, y+\theta\Delta y)}{f'_y(x+\theta\Delta x, y+\theta\Delta y)}, \tag{8}$$

where the point $(x + \theta\Delta x, y + \theta\Delta y) \in \Delta$. By virtue of the conditions of the theorem, in the closed rectangle $\overline{\Delta}$, and, consequently, also in the rectangle $\Delta \subset \overline{\Delta}$, the function f'_x is bounded ($|f'_x| \leqslant M$), and, by (7), the function f'_y is bounded below by a number $m > 0$, therefore, from (8) we obtain that

$$\left|\frac{\Delta y}{\Delta x}\right| \leqslant \frac{M}{m},$$

i.e. $\Delta y \to 0$ as $\Delta x \to 0$, which means that the function $y = \psi(x)$ is continuous at the point x. Since x is an arbitrary point of Δ^0, the function $\psi(x)$ is continuous in Δ^0.

Now, passing to the limit in (8) as $\Delta x \to 0$, we get (as was proved, Δy also tends to zero)

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\frac{f'_x(x, y)}{f'_y(x, y)} \quad (y = \psi(x)). \tag{9}$$

We have proved the existence of the derivative $\psi'(x)$ at the point x and the equality

$$\psi'(x) = -\frac{f'_x(x, \psi(x))}{f'_y(x, \psi(x))}. \tag{10}$$

The continuity of $\psi'(x)$ is directly implied by (10), since f'_x and f'_y are continuous in the rectangle Δ and the curve $y = \psi(x)$, whose continuity has been already proved, does not fall outside it.

Let us formulate a theorem, analogous to Theorem 1, for the case when an implicit function depends on n variables.

Theorem 1′. *Let there be given an equation*

$$f(\mathbf{x}, y) = f(x_1, \ldots, x_n, y) = 0, \tag{1'}$$

satisfying the following conditions.

The function f is defined in a neighbourhood Ω of a point $(\mathbf{x}^0, y^0) = (x_1^0, \ldots, x_n^0, y^0)$ belonging to the space R_{n+1} of points $(\mathbf{x}, y) = (x_1, \ldots, x_n, y)$ and is continuous there together with its first-order partial derivatives; the derivative with respect to y is different from zero and the function itself is equal to zero at $(\mathbf{x}^0, y^0)$:

$$f'_y(\mathbf{x}^0, y^0) = \left(\frac{\partial f}{\partial y}\right)_0 \neq 0, \quad f(\mathbf{x}^0, y^0) = 0. \tag{2'}$$

Let $\mathfrak{M}$ be the set of all points $(\mathbf{x}, y)$ satisfying equation (1′) (in particular, $(\mathbf{x}^0, y^0) \in \mathfrak{M}$).

Then for any $b_0 > 0$ there can be found a rectangle

$$\Delta = \{|x_j - x_j^0| < a,\ j = 1, \ldots, n,\ |y - y^0| < b\},\ b < b_0, \tag{3'}$$

lying within Ω, such that the set $\mathfrak{M}\Delta$ is described by a continuously differentiable (i.e. having continuous partial derivatives) function

$$y = \psi(\mathbf{x}) = \psi(x_1, \ldots, x_n), \quad \mathbf{x} \in \Delta^0, \tag{4'}$$

where

$$\Delta^0 = \{|x_j - x_j^0| < a,\ j = 1, \ldots, n\}. \tag{5'}$$

Partial derivatives of the function ψ are computed by the formula

$$\frac{\partial \psi}{\partial x_j} = -\frac{\partial f}{\partial x_j} \Big/ \frac{\partial f}{\partial y} \quad (j = 1, \ldots, n) \qquad (10')$$

If the function f (in the case of Theorems 1 and 1′) has continuous derivatives of a higher order l, then the implicit function also has derivatives of order l which can be found by differentiating formula (10) or (10′) l times.

Example. Let it be known that the function $f(x, y)$ considered in Theorem 1 has continuous partial derivatives of the second order. We shall proceed from equality (10). Differentiating it with respect to x, we obtain

$$\psi''(x) = -\frac{f'_y(f''_{x^2} + f''_{xy}\psi') - f'_x(f''_{xy} + f''_{y^2}\psi')}{(f'_y)^2}.$$

We have used the formula for differentiating a composite function.

Sec. 8.16. Tangent Plane and Normal

Let a surface S be represented by the equation

$$F(x, y, z) = 0 \qquad (1)$$

implicitly. We shall assume that $F(x_0, y_0, z_0) = 0$ and that in a certain neighbourhood of the point (x_0, y_0, z_0) the function F has continuous partial derivatives which are not all zero. Then

$$\operatorname{grad}_0 F = ((F'_x)_0, (F'_y)_0, (F'_z)_0) \neq 0. \qquad (2)$$

We shall write Φ_0 instead of $\Phi(x_0, y_0, z_0)$.

For the sake of definiteness, we suppose that $(F'_z)_0 \neq 0$. Then, on the basis of the theorem on an implicit function, there exists a neighbourhood of the point (x_0, y_0, z_0) in which the surface S is described explicitly by a continuously differentiable function $z = f(x, y)$. As we know, the equation of the tangent plane to the surface S at its point (x_0, y_0, z_0) has the form

$$z - z_0 = (f'_x)_0 (x - x_0) + (f'_y)_0 (y - y_0),$$

where

$$(f'_x)_0 = -(F'_x)_0/(F'_z)_0, \quad (f'_y)_0 = -(F'_y)_0/(F'_z)_0.$$

Hence, the equation of the tangent plane to S at its point (x_0, y_0, z_0) is finally written as follows:

$$(F'_x)_0 (x - x_0) + (F'_y)_0 (y - y_0) + (F'_z)_0 (z - z_0) = 0, \quad (3)$$

and the equation of the normal to S at the point (x_0, y_0, z_0) as:

$$\frac{x - x_0}{(F'_x)_0} = \frac{y - y_0}{(F'_y)_0} = \frac{z - z_0}{(F'_z)_0}. \quad (4)$$

The same equations (3), (4) are obtained if we assume that $(F'_x)_0 \neq 0$ or $(F'_y)_0 \neq 0$. In these cases in a neighbourhood of (x_0, y_0, z_0) the surface S is described explicitly by the respective equations

$$x = \varphi(y, z), \quad y = \psi(x, z).$$

We see that under condition (2) a certain piece of the surface S belonging to a sufficiently small neighbourhood of (x_0, y_0, z_0) has, at any of its points, a tangent plane which continuously changes as the point (x_0, y_0, z_0) continuously changes its position. Such a piece is called a *smooth piece of S*.

Quite another matter if $\operatorname{grad}_0 F = 0$. In this case we cannot guarantee that there exists a tangent plane to S at the point (x_0, y_0, z_0). It may exist and it may not.

Example. The equation

$$z^2 + y^2 - x^2 = 0 \quad (5)$$

defines a circular cone with the vertex at the origin and axis coinciding with the x-axis (Fig. 102).

The left-hand member of equation (5) has partial derivatives

$$F'_x = -2x, \ F'_y = 2y, \ F'_z = 2z;$$

which are not all zero, if the point $(x, y, z) \neq (0, 0, 0)$. At any such point, which will be denoted by (x_0, y_0, z_0), the tangent plane is defined by the equation

$$-x_0 (x - x_0) + y_0 (y - y_0) + z_0 (z - z_0) = 0.$$

And at the origin there is no tangent plane to our conic surface. In this case $\operatorname{grad}_0 F = 0$.

Points (x_0, y_0, z_0), lying on a surface S, at which $\operatorname{grad}_0 F = 0$ are called *singular points of the surface* S.

Let us consider a continuously differentiable function

$$u = f(x, y, z) \tag{6}$$

in some domain Ω of points (x, y, z). Let at the point $(x_0, y_0, z_0) \in \Omega$ its value be equal to the number A:

$$A = f(x_0, y_0, z_0).$$

If the partial derivatives of f at the point (x_0, y_0, z_0) are not all zero, then the equation $A = f(x, y, z)$ defines

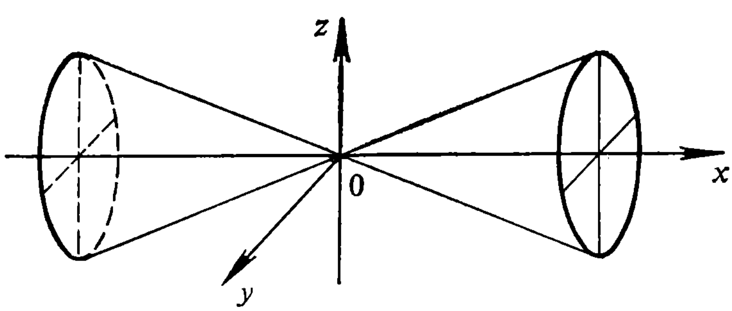

Fig. 102

in a neighbourhood of this point a certain smooth surface S called the *level surface of function* (6).

The tangent plane to S at the point (x_0, y_0, z_0) has the equation

$$\left(\frac{\partial f}{\partial x}\right)_0 (x-x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y-y_0) + \left(\frac{\partial f}{\partial z}\right)_0 (z-z_0) = 0.$$

The normal to S at the point (x_0, y_0, z_0), i.e. a straight line passing through this point perpendicular to the tangent plane, obviously, has the equation

$$\frac{X-x_0}{\left(\frac{\partial f}{\partial x}\right)_0} = \frac{Y-y_0}{\left(\frac{\partial f}{\partial y}\right)_0} = \frac{Z-z_0}{\left(\frac{\partial f}{\partial z}\right)_0}.$$

We see that the vector

$$\operatorname{grad} f_0 = \left(\left(\frac{\partial f}{\partial x}\right)_0, \left(\frac{\partial f}{\partial y}\right)_0, \left(\frac{\partial f}{\partial z}\right)_0\right)$$

is directed along the normal towards the surface S.

The equation $z = f(x, y)$, where the function f has continuous partial derivatives, defines a smooth surface S. Let us put $A = f(x_0, y_0)$. If at the point (x_0, y_0) the partial derivatives $\left(\frac{\partial f}{\partial x}\right)_0$, $\left(\frac{\partial f}{\partial y}\right)_0$ are not both zero, then the equation $A = f(x, y)$ defines some smooth curve Γ (*level curve* of the function $z = f(x, y)$) in a neighbourhood of this point.

The equation of the tangent to Γ at (x_0, y_0) has the form

$$\left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0) = 0.$$

The vector $\left(\left(\frac{\partial f}{\partial x}\right)_0, \left(\frac{\partial f}{\partial y}\right)_0\right)$ is directed along the normal towards Γ at the point (x_0, y_0).

Sec. 8.17. Systems of Functions Represented Implicitly

In Sec. 8.15 we considered the question of existence of continuous and differentiable implicit function defined by one equation.

Here we shall discuss a similar problem for a set of implicit functions $y_1, \ldots, y_m$ defined by a system of equations

$$\left.\begin{array}{l} f_1(x_1, \ldots, x_n;\ y_1, \ldots, y_m) = 0, \\ \dots\dots\dots\dots\dots\dots\dots \\ f_m(x_1, \ldots, x_n;\ y_1, \ldots, y_m) = 0. \end{array}\right\} \qquad (1)$$

Thus, we seek for the functions $y_1, \ldots, y_m$ of $x_1, \ldots, x_n$ as for the solutions of system (1) : $y_i = \psi_i(x_1, \ldots, x_n)$ $(i = 1, \ldots, m)$.

Let us clarify the conditions which ensure the existence of a solution of system (1) and differentiability of the functions y_i.

Theorem 1. *Let there be given a system of equations* (1) *satisfying the following properties.*

The functions f_j are defined in a ((n + m)-dimensional) neighbourhood Ω of the point $(\mathbf{x}^0, \mathbf{y}^0) = (x_1^0, \ldots, x_n^0, y_1^0, \ldots, y_m^0)$ belonging to the space R_{n+m} of points $(\mathbf{x}, \mathbf{y}) =$

$= (x_1, \ldots, x_n; y_1, \ldots, y_m)$ *and are continuous there together with their partial derivatives of the first order with the Jacobian (Jacobi's* determinant)*

$$\begin{vmatrix} \frac{\partial f_1}{\partial y_1} & \ldots & \frac{\partial f_1}{\partial y_m} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \frac{\partial f_m}{\partial y_1} & \ldots & \frac{\partial f_m}{\partial y_m} \end{vmatrix} = \left| \frac{\partial f_i}{\partial y_j} \right| = \frac{D(f_1, \ldots, f_m)}{D(y_1, \ldots, y_m)} \neq 0. \quad (2)$$

Besides, the point $(\mathbf{x}^0, \mathbf{y}^0)$ *satisfies system* (1).

Let $\mathfrak{M}$ *be the set of all points* $(\mathbf{x}, \mathbf{y})$ *satisfying system* (1) (*in particular,* $(\mathbf{x}^0, \mathbf{y}^0) \in \mathfrak{M}$).

Then for any $b_0 > 0$ *there is a rectangle*

$$\Delta = \{|x_j - x_j^0| < a \ (j = 1, \ldots, n),$$
$$|y_i - y_i^0| < b \ (i = 1, \ldots, m)\}, \ b < b_0 \quad (3)$$

belonging to Ω *such that the set* $\mathfrak{M}\Delta$ *is described by continuously differentiable functions*

$$y_i = \psi_i(\mathbf{x}) \quad (i = 1, \ldots, m), \ \mathbf{x} \in \Delta^0, \quad (4)$$

where

$$\Delta^0 = \{|x_j - x_j^0| < a, \quad j = 1, \ldots, n\}. \quad (5)$$

In other words, the rectangle Δ possesses the property that it is possible to define, in its projection Δ^0 on the coordinate subspace $(x_1, \ldots, x_n)$ continuously differentiable functions (4) satisfying equations (1), i.e.

$$f_j(\mathbf{x}, \psi_1(\mathbf{x}), \ldots, \psi_m(\mathbf{x})) \equiv 0, \quad \mathbf{x} \in \Delta^0 \ (j = 1, \ldots, m)$$

and the inequalities $|\psi_j(\mathbf{x}) - y_j^0| < b$. These functions are determined uniquely in the sense that the coordinates of any point $(\mathbf{x}, \mathbf{y}) \in \mathfrak{M}\Delta$ are connected by equations (4).

In particular, $y^0{}_j = \psi_j(\mathbf{x}^0)$ $(j = 1, \ldots, m)$ because $(\mathbf{x}^0, \mathbf{y}^0) \in \mathfrak{M}\Delta$.

Remark 1. It is allowable to assume that the rectangle Δ and its projection Δ^0 mentioned in the theorem are speci-

* Jacobi, Karl Gustav Jacob (1804-1851). German mathematician.

fied by inequalities

$$\Delta = \{|x_j - x_j^0| < a_j \quad (j = 1, \ldots, n);$$
$$|y_i - y_i^0| < b_i \quad (i = 1, \ldots, m)\}, \qquad (3^*)$$
$$\Delta^0 = \{|x_j - x_j^0| < a_j, \quad j = 1, \ldots, n\}, \quad (5^*)$$

where, generally speaking, a_j and b_i are different numbers. For if the theorem holds for rectangle (3) with some a_j and b_i, then, setting $b = \min b_i$ and taking into account the continuity of the functions ψ_i, we can find a number $a < a_j$ $(j = 1, \ldots, n)$ such that the points

$$(\mathbf{x}, \psi_1(\mathbf{x}), \ldots, \psi_m(\mathbf{x})) \text{ with } \mathbf{x} \in \{|x_j - x_j^0| < a, \quad j = 1, \ldots, n\}$$

are in rectangle (3).

However, it should be noted that in the general case it may be impossible to choose a and b in (3) so that they are equal; this is readily confirmed by the example of the equation $F(x, y) = y - x^2 = 0$ considered in a neighbourhood of the point $x_0 = y_0 = 0$.

Let us prove Theorem 1 only for a particular case of two equations

$$\left.\begin{aligned} f_1(x_1, x_2, y_1, y_2) = 0, \\ f_2(x_1, x_2, y_1, y_2) = 0. \end{aligned}\right\} \qquad (1')$$

We have to prove that if the functions f_1 and f_2 are continuously differentiable in some neighbourhood of the point $M^0 = (x_1^0, x_2^0, y_1^0, y_2^0) \in R_4$, satisfying equations (1′) and the Jacobian

$$\begin{vmatrix} \dfrac{\partial f_1}{\partial y_1} & \dfrac{\partial f_1}{\partial y_2} \\ \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} \end{vmatrix} \neq 0 \qquad (2')$$

at M^0, then for any $b_0 > 0$ there is a rectangle

$$\Delta = \{|x_1 - x_1^0| < a, \ |x_2 - x_2^0| < a, \ |y - y_1^0| < b, \ |y_2 - y_2^0| < b\} \qquad (3')$$
$$(b < b_0),$$

belonging to the indicated neighbourhood and there exist continuously differentiable functions

$$\left.\begin{aligned} y_1 = \psi_1(x_1, x_2), \\ y_2 = \psi_2(x_1, x_2), \end{aligned}\right\} (x_1, x_2) \in \Delta^0, \qquad (4')$$

defined on its projection

$$\Delta^0 = \{|x_1 - x_1^0| < a, \ |x_2 - x_2^0| < a\}, \tag{5'}$$

such that they satisfy equations (1) and possess the following properties:

$$y_1^0 = \psi_1(x_1^0, x_2^0), \quad y_2^0 = \psi_2(x_1^0, x_2^0).$$

In this case for $(x_1, x_2) \in \Delta^0$

$$(x_1, x_2, \psi_1(x_1, x_2), \quad \psi_2(x_1, x_2)) \in \Delta. \tag{6}$$

The functions ψ_1 and ψ_2 are unique; they describe all the solutions of equations (1′) in the rectangle Δ. In other words, if some point $(x_1, x_2, y_1, y_2) \in \Delta$ satisfies equations (1′), then its coordinates are connected with relationships (4′).

From the fact that Jacobian (2′) is not equal to zero at M^0, it follows that one of its elements is not equal to zero at M^0. Without loss of generality, we assume that

$$\frac{\partial f_1}{\partial y_1} \neq 0. \tag{7}$$

This can always be obtained by renumbering f_1, f_2 and y_1, y_2, if necessary.

Since, by the hypothesis, the partial derivatives of f_1 and f_2 are continuous, there exists a sufficiently small neighbourhood of the point M^0 in which not only Jacobian (2′), but also the derivative $\frac{df_1}{\partial y_1}$ are not equal to zero. But then for the first equation of (1′), if it is considered with respect to the unknown function y_1 of (x_1, x_2, y_2), the conditions of Theorem 1′ from Sec. 8.15 are fulfilled. Therefore, for any b_0 there exists a rectangle

$$\Delta_1 = \{|x_1 - x_1^0| < \alpha, \ |x_2 - x_2^0| < \alpha, \ |y_2 - y_2^0| < \beta, \\ |y_1 - y_1^0| < \gamma\} \tag{8}$$

$$(\gamma < b_0)$$

and a continuously differentiable function

$$y_1 = \varphi(x_1, x_2, y_2), \tag{9}$$

$$(x_1, x_2, y_2) \in \Delta_1^0 = \{|x_1 - x_1^0| < \alpha, \ |x_2 - x_2^0| < \alpha, \\ |y_2 - y_2^0| < \beta\},$$

satisfying the first equation of (1′):

$$f_1(x_1, x_2, \varphi(x_1, x_2, y_2), y_2) = 0, \tag{10}$$

where

$$(x_1, x_2, y_2) \in \Delta_1^0, \quad (x_1, x_2, \varphi(x_1, x_2, y_2), y_2) \in \Delta_1. \tag{11}$$

The function φ is unique in the sense that any point (x_1, x_2, y_1, y_2) belonging to Δ_1 and satisfying the first equation of (1′) has coordinates related by equation (9); in particular,

$$y_1^0 = \varphi(x_1^0, x_2^0, y_2^0). \tag{12}$$

Remark 2. Note that in (8) we could set $\alpha = \beta$ at the first stage of reasoning. But for our further aims, the numbers α and β must be somewhat reduced, generally speaking, not proportionally. The reduced α and β are suitable for the first stage of reasoning as well.

Thus, we have obtained identity (10) which is true for any independent $(x_1, x_2, y_2) \in \Delta_1^0$. But this identity remains true for the supposition that y_2 is any continuously differentiable function $y_2 = \psi_2(x_1, x_2)$, but such that

$$(x_1, x_2, \psi_2(x_1, x_2)) \in \Delta_1^0. \tag{13}$$

But there is an infinite number of such functions ψ_2. Our aim is to choose one of them such that the functions

$$\left.\begin{aligned} y_1 &= \varphi(x_1, x_2, \psi_2(x_1, x_2)) = \psi_1(x_1, x_2), \\ y_2 &= \psi_2(x_1, x_2) \end{aligned}\right\} \tag{14}$$

identically satisfy the second equation (1′). The first equation (1′) is already satisfied by them.

Let us now substitute the found function φ into the second equation (1′):

$$f_2(x_1, x_2, \varphi(x_1, x_2, y_2), y_2) = 0 \tag{15}$$

and seek for the function y_2 of (x_1, x_2) which satisfies (15). Here we set

$$\Phi(x_1, x_2, y_2) = f_2(x_1, x_2, \varphi(x_1, x_2, y_2), y_2).$$

The function Φ is continuously differentiable for any (x_1, x_2, y_2) belonging to Δ_1^0 (see (11)). It satisfies the equalities

$$\Phi(x_1^0, x_2^0, y_2^0) = f_2(x_1^0, x_2^0, \varphi(x_1^0, x_2^0, y_2^0), y_2^0) = f_2(x_1^0, x_2^0, y_1^0, y_2^0) = 0$$

(see the conditions of the theorem and (12)). Besides,

$$\frac{\partial \Phi}{\partial y_2} \neq 0.$$

Indeed, for the points $(x_1, x_2, y_2) \in \Delta_1^0$ (the explanations are given below)

$$\frac{\partial \Phi}{\partial y_2} = \frac{\partial f_2}{\partial y_1}\frac{\partial \varphi}{\partial y_2} + \frac{\partial f_2}{\partial y_2} = \frac{\partial f_2}{\partial y_1}\left(-\frac{\partial f_1}{\partial y_2}\Big/\frac{\partial f_1}{\partial y_1}\right) + \frac{\partial f_2}{\partial y_2}$$

$$= \left(\frac{\partial f_1}{\partial y_1}\frac{\partial f_2}{\partial y_2} - \frac{\partial f_2}{\partial y_1}\frac{\partial f_1}{\partial y_2}\right)\Big/\frac{\partial f_1}{\partial y_1} = \begin{vmatrix} \dfrac{\partial f_1}{\partial y_1} & \dfrac{\partial f_1}{\partial y_2} \\ \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} \end{vmatrix} \Big/ \frac{\partial f_1}{\partial y_1} \neq 0. \tag{16}$$

In the first equality (16) we have applied the rule for the derivative of a composite function, in the second it should be taken into account that, according to (10),

$$\frac{\partial \varphi}{\partial y_2} = -\frac{\partial f_1}{\partial y_2}\Big/\frac{\partial f_1}{\partial y_1}.$$

Of course, the above written partial derivatives of f_1 and f_2 are regarded as computed at the points $x_1 = x_1$, $x_2 = x_2$, $y_1 = \varphi(x_1, x_2, y_2)$, $y_2 = y_2$. In relation (16), we take into consideration (2′) and (7).

We see that the left-hand member of equation (15) satisfy all the conditions of Theorem 1′ from Sec. 8.15. Therefore in the rectangle Δ_1^0 (see (9)) we can find, generally speaking, a smaller rectangle which will be denoted again by Δ_1^0 (see Remark 2 above) and a continuously differentiable function

$$y_2 = \psi_2(x_1, x_2),$$
$$(x_1, x_2) \in \Delta^0 = \{|x_1 - x_1^0| < \alpha, \ |x_2 - x_2^0| < \alpha\},$$
$$(x_1, x_2, \varphi_2(x_1, x_2)) \in \Delta_1^0,$$

satisfying equation (15):

$$\Phi(x_1, x_2, \psi_2(x_1, x_2)) = f_2(x_1, x_2, \varphi(x_1, x_2, \psi_2(x_1, x_2)),$$
$$\psi_2(x_1, x_2)) = f_2(x_1, x_2, \psi_1(x_1, x_2), \psi_2(x_1, x_2)) = 0,$$
$$(x_1, x_2) \in \Delta^0, \ (x_1, x_2, \psi_2(x_1, x_2)) \in \Delta_1^0, \ \psi_2(x_1^0, x_2^0) = y_2^0.$$

In addition, any point (x_1, x_2, y_2) belonging to Δ_1^0 and satisfying equation (15) has coordinates connected by the equality $y_2 = \psi_2(x_1, x_2)$. But then the relation

$$(x_1, x_2, \psi_1(x_1, x_2), \quad \psi_2(x_1, x_2)) \in \Delta_1$$

is also fulfilled (see (11)).

Hence, we have proved the existence of the continuously differentiable functions

$$\left.\begin{aligned} y_1 &= \psi_1(x_1, x_2), \\ y_2 &= \psi_2(x_1, x_2), \end{aligned}\right\} \quad (x_1, x_2) \in \Delta^\circ,$$

satisfying both equations (1′) so that

$$y_1^0 = \psi_1(x_1^0, x_2^0), \quad y_1^0 = \psi_2(x_1^0, x_2^0),$$

where

$$(x_1, x_2, \psi_1(x_1, x_2), \psi_2(x_1, x_2)) \in \Delta_1 \tag{17}$$

and any point (x_1, x_2, y_1, y_2) belonging to Δ_1 and satisfying equations (1′) has the form as in (17).

The transition from Δ_1 to Δ can be realized with the aid of Remark 1.

Remark 3. Let us consider the method for finding the partial derivatives $\dfrac{\partial y_j}{\partial x_k}$. Let all the conditions of Theorem 1 be fulfilled. Then, substituting the functions $y_j = \psi_j(x)$ into (1), we obtain a system of identities:

$$\left.\begin{aligned} &f_1(x, \psi_1(x), \ldots, \psi_m(x)) \equiv 0 \\ &\cdots\cdots\cdots\cdots\cdots \\ &f_m(x, \psi_1(x), \ldots, \psi_m(x)) \equiv 0. \end{aligned}\right\}. \tag{18}$$

Differentiating each identity of system (18) as composite functions, we obtain

$$\left.\begin{aligned} &\frac{\partial f_1}{\partial x_k}+\frac{\partial f_1}{\partial y_1}\frac{\partial \psi_1}{\partial x_k}+\cdots+\frac{\partial f_1}{\partial y_m}\frac{\partial \psi_m}{\partial x_k}\equiv 0\\ &\quad\cdots\cdots\cdots\cdots\cdots\cdots\\ &\frac{\partial f_m}{\partial x_k}+\frac{\partial f_m}{\partial y_1}\frac{\partial \psi_1}{\partial x_k}+\cdots+\frac{\partial f_m}{\partial y_m}\frac{\partial \psi_m}{\partial x_k}\equiv 0.\end{aligned}\right\} \tag{19}$$

System (19) is linear with respect to the unknown derivatives $\frac{\partial \psi_j}{\partial x_k}$ $(j=1, \ldots, m)$ whose determinant is the Jacobian

$$\frac{D(f_1, \ldots, f_m)}{D(y_1, \ldots, y_m)}\neq 0.$$

Therefore this system has the unique solution:

$$\frac{\partial y_1}{\partial x_k}=-\frac{D(f_1, \ldots, f_m)}{D(x_k, y_2, \ldots, y_m)}\Big/\frac{D(f_1, \ldots, f_m)}{D(y_1, \ldots, y_m)}, \ldots,$$

$$\frac{\partial y_m}{\partial x_k}=-\frac{D(f_1, \ldots, f_m)}{D(y_1, \ldots, y_{m-1}, x_k)}\Big/\frac{D(f_1, \ldots, f_m)}{D(y_1, \ldots, y_m)}.$$

Sec. 8.18. Mappings

Let there be given a system of continuously differentiable functions

$$y_j=\varphi_j(\mathbf{x})=\varphi_j(x_1, \ldots, x_n),\ \mathbf{x}\in\Omega\ (j=1, \ldots, m), \tag{1}$$

where Ω is an open set of points $\mathbf{x}=(x_1, \ldots, x_n)$. We shall say that system (1) specifies a *continuously differentiable mapping*

$$y=A\mathbf{x}, \quad \mathbf{x}\in\Omega, \tag{1'}$$

of the set Ω on the corresponding set Ω' of points $\mathbf{y}= = (y_1, \ldots, y_m)$. We shall also write $\Omega'=A(\Omega)$ and call Ω' the *image* of Ω and Ω the *preimage* (or the inverse image) or Ω' (under the mapping A).

Along with A, let us consider another continuously differentiable mapping B of an open set Λ of points $\mathbf{y}$ on a set of points $\mathbf{z}=(z_1, \ldots, z_m)$ determined by some functions

$$z_j=\psi_j(\mathbf{y})=\psi_j(y_1, \ldots, y_m),\ \mathbf{y}\in\Lambda\ (j=1, \ldots, m).$$

Thus $\mathbf{z}=B\mathbf{y}$, $\mathbf{y}\in\Lambda$.

Remark. Note that if $\mathbf{x}^0 \in \Omega$ and $\mathbf{y}^0 = A\mathbf{x}^0 \in \Lambda$, then by the continuity of A, there is a neighbourhood V_{x^0} of the point $\mathbf{x}^0$ whose image under the mapping A belongs to Λ. Reducing Ω by setting $\Omega = V_{x^0}$, we then obtain $\Omega' \subset \Lambda$.

If $\Omega' \subset \Lambda$, then we can consider the composite continuously differentiable mapping $\mathbf{z} = BA\mathbf{x}$, $\mathbf{x} \in \Omega$, specified by the equalities $z_j = \psi_j(\varphi_1(\mathbf{x}), \ldots, \varphi_m(\mathbf{x}))$, $\mathbf{x} \in \Omega$ $(j = 1, \ldots, m)$.

The Jacobians of the mappings A, B, and BA are connected by the important relations

$$\frac{D(z_1, \ldots, z_m)}{D(x_1, \ldots, x_m)} = \left|\frac{\partial z_i}{\partial x_j}\right| = \left|\sum_{s=1}^{m} \frac{\partial z_i}{\partial y_s}\frac{\partial y_s}{\partial x_j}\right| = \left|\frac{\partial z_i}{\partial y_s}\right| \cdot \left|\frac{\partial y_s}{\partial x_j}\right|$$

$$= \frac{D(z_1, \ldots, z_m)\, D(y_1, \ldots, y_m)}{D(y_1, \ldots, y_m)\, D(x_1, \ldots, x_m)}. \tag{2}$$

whose proof, as we see, is based on the application of the differentiation formula for a composite function and on the multiplication rule for determinants.

In particular, if B is the inverse (the inverse mapping) of A which maps the set $A(\Omega)$ on the set of points $\mathbf{x} \in \Omega$, i.e., $\mathbf{x} = BA\mathbf{x}$, $\mathbf{x} \in \Omega$, is the identity mapping, then, since its Jacobian is equal to 1, we receive the formula

$$1 = \frac{D(x_1, \ldots, x_m)}{D(y_1, \ldots, y_m)} \cdot \frac{D(y_1, \ldots, y_m)}{D(x_1, \ldots, x_m)}, \quad \mathbf{x} \in \Omega. \tag{3}$$

Now we shall suppose that the Jacobian of the continuously differentiable mapping $y = A\mathbf{x}$ specified by equalities (1) is different from zero throughout the open set Ω:

$$\frac{D(y_1, \ldots, y_m)}{D(x_1, \ldots, x_m)} \neq 0, \quad \mathbf{x} \in \Omega,$$

Let us give (without proof) the following properties:

(1) $\Omega' = A(\Omega)$ is an open set (together with Ω!);

(2) if Ω is a domain, then Ω' is also a domain;

(3) the mapping A is locally one-to-one, that is, for any point $\mathbf{x}^0 \in \Omega$, there is a sphere $V \subset \Omega$ with centre at that point such that the mapping A, considered only on V, is one-to-one.

Property (3) expresses the fact that the mapping A is only locally one-to-one, because, generally speaking, it may not be one-to-one throughout Ω. For instance, the transformation $x = \rho\cos\theta$, $y = \rho\sin\theta$ from polar coordinates ρ, θ of the points of the xy-plane into the Cartesian coordinates x, y is continuously

differentiable and has a nonzero Jacobian equal to ρ for $\rho > 0$ and any θ. It maps the points (ρ, θ) $(\rho > 0, -\infty < \theta < \infty)$ of the (ρ, θ)-plane distinct from the origin onto the points of the xy-plane in a locally one-to-one manner. However, although to each such point (x, y) there corresponds a single value of ρ, the number of the corresponding values of θ which differ from each other by $2k\pi$ $(k = \pm 1, \pm 2, \ldots)$.

Sec. 8.19. Conditional (Relative) Extremum

Let us consider the function $u = F(x, y) = x^2 + y^2$ in the space R_2. It is easily seen that from the geometrical point of view this function represents the square of the distance of the point $P = (x, y)$ from the origin of a rectangular system of coordinates (x, y). The given function has no greatest value in R_2; but if it is considered only for the points (x, y) of the ellipse $G(x, y) \equiv \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ $(b > a)$, then it becomes evident that our function attains its greatest value at the points $P_0 = (0, b)$ and $P_1 = (0, -b)$ (see Fig. 103).

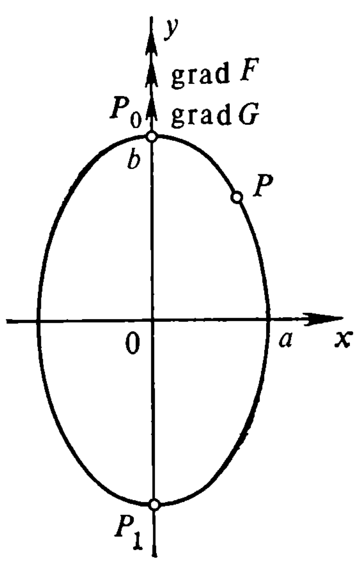

Fig. 103

Thus, a function $u = F(P)$, considered in the entire plane R_2, has no greatest value, but the same function attains its greatest value (twice), provided the point P is found on the ellipse.

This situation leads us to the problem of finding the extremum of a function under the condition that its arguments satisfy some additional restrictions.

So, let there be given a function

$$u = F(P) = F(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

of $n + m$ variables. It is required to find the extremum of the function $F(P)$ under the condition that the variables

x_i, y_k are related by m equalities

$$\left.\begin{array}{l} G_1(x_1, \ldots, x_n, y_1, \ldots, y_m)=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ G_m(x_1, \ldots, x_n, y_1, \ldots, y_m)=0, \end{array}\right\} \qquad (1)$$

which are usually called *coupling equations* (or *constraint equations*, or, simply, *constraints*).

System of equations (1) defines in the space R_{n+m}, generally speaking, a certain set which will be called a *surface*.

Definition. We shall say that a point $P^0 = (x_1^0, \ldots, x_n^0, y_1^0, \ldots, y_m^0)$ satisfying equations (1) is a *point of local conditional* (*relative*) *maximum* (*minimum*) if in R_{n+m} there exists a neighbourhood of the point P^0 such that for any P from this neighbourhood satisfying coupling equations (1) the following inequality is fulfilled:

$$F(P) \leqslant F(P^0) \quad (F(P) \geqslant F(P^0)).$$

Points of local conditional maxima and minima are referred to as *points of local conditional* (*relative*) *extremum*.

In the above considered example the point $P_0 = (0, b)$ is a point of local maximum, since for all the points P, lying on the ellipse, $u(P) \leqslant u(P_0)$.

We shall begin with establishing necessary conditions for a given point P^0 to be a point of local conditional extremum. Let P^0 be a point of conditional extremum and let the functions $G_1, \ldots, G_m$ have continuous partial derivatives and the Jacobian

$$\frac{D(G_1, \ldots, G_m)}{D(y_1, \ldots, y_m)} \neq 0 \qquad (2)$$

in a neighbourhood of this point.

As we know, system (1) is solvable with respect to the variables $y_1, \ldots, y_m$ in some neighbourhood of the point P^0:

$$y_i = \varphi_i(x_1, \ldots, x_n) \quad (i = 1, \ldots, m),$$

where the functions $\varphi_i(x_1, \ldots, x_n)$ have continuous partial derivatives at the point $M^0 = (x_1^0, \ldots, x_n^0)$.

Substituting the functions φ_i into F, we obtain that F is a function only of n variables $x_1, \ldots, x_n$, independent

of one another:

$$F(x_1, \ldots, x_n, \varphi_1(x_1, \ldots, x_n), \ldots, \varphi_m(x_1, \ldots, x_n)) \equiv \Phi(x_1, \ldots, x_n). \qquad (3)$$

It is obvious that if F attains a local conditional extremum at the point P^0, then $\Phi(x_1, \ldots, x_n)$ attains an ordinary local extremum, or, as we say, local unconditional extremum at the point $M^0 = (x_1^0, \ldots, x_n^0)$, and vice versa.

But then, as we know, there must be fulfilled the equalities

$$\frac{\partial \Phi(M^0)}{\partial x_i} = 0 \ (i = 1, \ldots, n) \ \text{ or } \ d\Phi(M^0) = \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i} dx_i = 0, \qquad (4)$$

where dx_i $(i = 1, \ldots, n)$ are differentials of the independent variables.

The point $P^0 = (x_1^0, \ldots, x_n^0, y_1^0, \ldots, y_m^0)$ for which, by virtue of (1) (or (3)), (4) is fulfilled will be called a *stationary point of the function F under constraints* (1).

We have proved that *for a point* $P^0 = (x_1^0, \ldots, x_n^0, y_1^0, \ldots, y_m^0)$ *to be a point of local conditional extremum, it is necessary that it be a stationary point of F under constraints* (1).

Our further considerations refer to the question of how to find the indicated stationary point without solving system (1) with respect to the variables $y_1, \ldots, y_m$, although the existence of functions $\varphi_1, \ldots, \varphi_m$ will be supposed. We shall write $(F)_0$, $(\varphi_i)_0$ instead of $F(P^0)$, $\varphi_i(M^0)$.

By the invariance of the form of the first differential conditions (4) are equivalent to the conditions

$$d\Phi(M^0) = dF(P^0) = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i}\right)_0 dx_i + \sum_{k=1}^{m} \left(\frac{\partial F}{\partial y_k}\right)_0 dy_k = 0, \qquad (5)$$

where the dependent differentials $dy_1, \ldots, dy_m$ entering into dF are respectively equal to

$$dy_k = \sum_{i=1}^{n} \left(\frac{\partial \varphi_k}{\partial x_i}\right)_0 dx_i \quad (k = 1, \ldots, m).$$

These differentials and the independent differentials $dx_1, \ldots, dx_n$ are connected by the relations

$$dG_i(P^0) = \sum_{j=1}^{n} \left(\frac{\partial G_i}{\partial x_j}\right)_0 dx_j + \sum_{k=1}^{m} \left(\frac{\partial G_i}{\partial y_k}\right)_0 dy_k = 0 \quad (i = 1, \ldots, m), \tag{6}$$

which are obtained from the coupling equations.

Thus, a stationary point of F under constraints (1) may also be defined as a point $P^0 = (x_1^0, \ldots, x_n^0, y_1^0, \ldots y_m^0)$, satisfying (1), such that for it equalities (5) are fulfilled for all $dx_1, \ldots, dx_n, dy_1, \ldots, dy_m$, for which equalities (6) fake place.

Let us introduce the $(n+m)$-dimensional vectors

$$\operatorname{grad}_0 G_j = \operatorname{grad} G_j(P^0) = \left(\left(\frac{\partial G_j}{\partial x_1}\right)_0, \ldots, \left(\frac{\partial G_j}{\partial x_n}\right)_0, \left(\frac{\partial G_j}{\partial y_1}\right)_0, \ldots, \left(\frac{\partial G_j}{\partial y_m}\right)_0\right) \quad (j = 1, \ldots, m),$$

$$\operatorname{tgrad}_0 F = \operatorname{grad} F(P^0) = \left(\left(\frac{\partial F}{\partial x_1}\right)_0, \ldots, \left(\frac{\partial F}{\partial x_n}\right)_0, \left(\frac{\partial F}{\partial y_1}\right)_0, \ldots, \left(\frac{\partial F}{\partial y_m}\right)_0\right),$$

$$d\mathbf{z} = (dx_1, \ldots, dx_n, dy_1, \ldots, dy_m).$$

In terms of these vectors equations (5) and (6) can be written as scalar products

$$(\operatorname{grad}_0 F, d\mathbf{z}) = 0, \tag{5'}$$

$$(\operatorname{grad}_0 G_j, d\mathbf{z}) = 0 \quad (j = 1, \ldots m). \tag{6'}$$

We have obtained that the point $P^0 = (x_1^0, \ldots, x_n^0, y_1^0, \ldots, y_m^0)$ is a stationary point under constraints (1) if

and only if it satisfies equations (1) and if from the fact that some vector $d\mathbf{z}$ is orthogonal to the gradients $\operatorname{grad}_0 G_1, \ldots, \operatorname{grad}_0 G_m$ it follows that it is orthogonal to $\operatorname{grad}_0 F$. But in such a case (the explanations are given below) there exists a unique system of numbers $\lambda_1, \ldots, \lambda_m$ such that

$$\operatorname{grad}_0 F = \sum_{k=1}^{m} \lambda_k \operatorname{grad}_0 G_k. \tag{7}$$

The converse is also true. If it is known that for some numbers $\lambda_1, \ldots, \lambda_m$ $\operatorname{grad}_0 F$ can be represented in form (7), i.e. in the form of a linear combination of the gradients $\operatorname{grad}_0 G_k$ ($k = 1, \ldots, m$), hence it immediately follows that as soon as some vector $d\mathbf{z}$ is orthogonal to the gradients $\operatorname{grad}_0 G_k$, it is automatically orthogonal to $\operatorname{grad}_0 F$.

It is quite easy to verify the correctness of the inverse: from (7) and (6′) it follows that

$$(\operatorname{grad}_0 F, d\mathbf{z}) = \left(\sum_{k=1}^{m} \lambda_k \operatorname{grad}_0 G_k, d\mathbf{z}\right)$$

$$= \sum_{k=1}^{m} \lambda_k (\operatorname{grad}_0 G_k, d\mathbf{z}) = \sum_{k=1}^{m} \lambda_k \cdot 0 = 0.$$

As to the direct statement, we should like to refer to the theorem from linear algebra*. Nevertheless, let us give some explanations.

Let L be a linear subspace R_{n+m} stretched on the vectors $\operatorname{grad}_0 G_j$ ($j = 1, \ldots, m$), i.e. the set of linear combinations of form (7) corresponding to all possible systems of numbers $\lambda_1, \ldots, \lambda_m$. And let us introduce a subspace L' of vectors $d\mathbf{z}$, orthogonal to L, i.e. L' consists of all the vectors $d\mathbf{z}$, orthogonal to L, or, which is the same, orthogonal to the vectors $\operatorname{grad}_0 G_j$ ($j = 1, \ldots, m$). If L' is orthogonal to L**, then also conversely, L is orthogonal to L', i.e. L consists of all the vectors orthogonal to L'. As

* See our book *Fundamentals of Linear Algebra and Analytical Geometry*, Sec. 19, Theorems 1, 2 and Corollary 1.

** See the same book, Sec. 19, Theorem 1.

was stated, at the stationary point P^0 the gradient F is orthogonal to all the vectors dz which are orthogonal to the gradient G_j, i.e. the gradient F is orthogonal to L'. But then, according to the indicated theorem, gradient F belongs to L and is thus a certain linear combination of gradients G_j, that is a unique linear combination, since the gradients G_j $(j = 1, \ldots, m)$ form a linearly independent system in R_{n+m}. The matter is that the matrix formed from the partial derivatives of the functions G_j

$$\left\| \begin{matrix} \frac{\partial G_1}{\partial x_1} \cdots \frac{\partial G_1}{\partial x_n} \frac{\partial G_1}{\partial y_1} \cdots \frac{\partial G_1}{\partial y_m} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial G_m}{\partial x_1} \cdots \frac{\partial G_m}{\partial x_n} \frac{\partial G_m}{\partial y_1} \cdots \frac{\partial G_m}{\partial y_m} \end{matrix} \right\| \tag{8}$$

has in a neighbourhood of the point P^0 rank m, because we supposed that condition (2) is true, but then the rows of this matrix define the vectors (gradients) which form a linearly independent system*.

It follows from the aforesaid that a stationary point of F under constraints (1) can also be defined as follows: *this is a point* $P^0 = (x_1^0, \ldots, x_n^0, y_1^0, \ldots, y_m^0)$ *satisfying equations* (1) *for which the gradient* F *is a linear combination of gradients* G_j $(j = 1, \ldots, m)$

$$\operatorname{grad}_0 F = \sum_{j=1}^{m} \lambda_j \operatorname{grad}_0 G_j. \tag{7}$$

We may also state the definition of a stationary point as follows: *for a point*

$$P^0 = (x_1^0, \ldots, x_n^0, y_1^0, \ldots, y_m^0)$$

to be stationary for the function F *under constraints* (1), *it is necessary and sufficient that there be for it numbers* $\lambda_1, \ldots, \lambda_m$ *for which equality* (7) *is fulfilled.*

Since the rank of matrix (8) at the point P^0 is equal to m, to every stationary point there corresponds a unique system of numbers $\lambda_1, \ldots, \lambda_m$ for which equality (7) takes

---

* See the above mentioned book, Sec. 13.

place. Equality (7) is equivalent to the following:

$$\operatorname{grad}_0 [F - \sum_{j=1}^{m} \lambda_j G_j] = 0. \tag{9}$$

The function

$$L(P, \lambda) = F(P) - \sum_{j=1}^{m} \lambda_j G_j(P), \quad \lambda = (\lambda_1, \ldots, \lambda_m),$$

is called the *Lagrangian function*, and the numbers λ_j are termed *Lagrange multipliers*.

Let us rewrite conditions (9) in an expanded form:

$$\left.\begin{aligned} \frac{\partial L}{\partial x_j} &= \frac{\partial F}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial G_i}{\partial x_j} = 0 \quad (j = 1, \ldots, n), \\ \frac{\partial L}{\partial y_k} &= \frac{\partial F}{\partial y_k} - \sum_{i=1}^{m} \lambda_i \frac{\partial G_i}{\partial y_k} = 0 \quad (k = 1, \ldots, m). \end{aligned}\right\} \tag{9'}$$

The problem of finding the stationary points of F subject to constraints (1) has been reduced to solving a system consisting of equations (1) and (9′).

Let us sum up the aforesaid.

In order to find a stationary point

$$P^0 = (x_1^0, \ldots, x_n^0, y_1^0, \ldots, y_m^0)$$

of a function F subject to constraints, it is necessary to form the Lagrangian function and a system of equations (9′) and to solve this system together with constraint equations (1). All in all we shall have $n + 2m$ equations in $n + 2m$ unknowns $x_1, \ldots, x_n, y_1, \ldots, y_m, \lambda_1, \ldots$ $\ldots, \lambda_m$. In this case the solution of the system with respect to x_i and y_j will yield the point $(x_1^0, \ldots, x_n^0, y_1^0, \ldots, y_m^0)$ which will be a stationary point. The point of local conditional extremum are found among the stationary points. The question of whether a stationary point P^0 is in fact a point of conditional extremum is conveniently clarified by considering the second differential of the Lagrangian function. In determining the sign of $d^2L(P^0, \lambda)$, we should take into account that the differentials dy_k depend on the differentials dx_i.

Example. Let in the xy-plane there be given a figure bounded by the coordinate axes and the parabola $y + x^2 - 3 = 0$ $(0 \leqslant x \leqslant \sqrt{3})$. It is required to inscribe into this figure a rectangle whose sides are parallel to the coordinate axes and one of its vertices $M = (x, y)$ lies on the parabola so that the area of the inscribed rectangle is the greatest (Fig. 104).

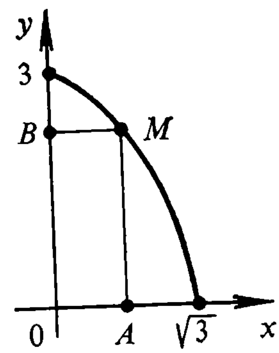

Fig. 104

Solution. Let x and y be the coordinates of the vertex M. Then the area of the rectangle will be $S = xy$. Further, since the point M lies on the parabola, its coordinates must satisfy the equation of the parabola: $y + x^2 - 3 = 0$. Hence, we have to investigate the function $S = xy$ for a conditional extremum under the constraint $y + x^2 - 3 = 0$. Let us introduce the Lagrangian function $L(x, y, \lambda) = xy - \lambda(y + x^2 - 3)$. The stationary points are found from the equations

$$\begin{cases} \dfrac{\partial L}{\partial x} = y - 2\lambda x = 0, \\ \dfrac{\partial L}{\partial y} = x - \lambda = 0, \\ y + x^2 - 3 = 0. \end{cases}$$

Solving this system, we find that $x = 1$, $y = 2$, $\lambda = 1$. Thus, the point $(1, 2)$ is a stationary one and there corresponds to it the Lagrange multiplier $\lambda = 1$. Let us now investigate at the obtained stationary point the second differential of the Lagrangian function

$$L(x, y, 1) = xy - y - x^2 + 3.$$

We have

$$d^2L(x, y, 1) = L''_{x^2}\,dx^2 + 2L''_{xy}\,dx\,dy + L''_{y^2}\,dy^2 = 2dx\,(dy - dx).$$

If dx and dy are regarded as differentials of independent variables, then $d^2L(x, y, 1)$ is not definite, as far as sign

is concerned. But it is seen from the constraint equation, that $dy = -2x\,dx$, at the point (1, 2) $dy = -2dx$. Hence,

$$d^2L\,(1,\ 2,\ 1) = -6dx^2 < 0 \quad (dx \neq 0),$$

and, consequently, the increment of the function $L\,(x, y, \lambda)$ at the point $x = 1$, $y = 2$, $\lambda = 1$ corresponding to the increment of x equal to $dx \neq 0$ is less than zero ($\Delta L\,(1,\ 2,\ 1) < 0$). This means that the function $S = xy$ attains a local conditional maximum at the point (1, 2), since on the parabola $y + x^2 - 3 = 0$, $\Delta S = \Delta L$.

Thus, of all the rectangles of the indicated form, the rectangle with sides $OA = 1$, $OB = 2$ has the greatest area.

CHAPTER 9

SERIES

Sec. 9.1. The Notion of a Series

An expression

$$u_0 + u_1 + u_2 + \dots, \tag{1}$$

where the numbers u_k (*the terms of the series*), which are complex in the general case, depend on the index $k = 0$, $1, 2, \dots$, is called a (number) *series*. Here we consider this expression purely formally and do not assign any number to it, since the ordinary addition of an infinite number of terms does not make sense. Series (1) can also be written in the following way:

$$\sum_{k=0}^{\infty} u_k = \sum_{0}^{\infty} u_k. \tag{2}$$

This brief notation, which is also purely formal, often proves to be more convenient than (1).

The number

$$S_n = u_0 + u_1 + \dots + u_n \qquad (n = 0, 1, \dots)$$

is called the *nth partial sum of series* (1).

Series (1) is said to be convergent if the limit

$$\lim_{n\to\infty} S_n = S$$

exists. In this case we write

$$S = u_0 + u_1 + u_2 + \dots = \sum_{k=0}^{\infty} u_k \tag{3}$$

and call S the *sum of the series*; in this very sense we assign to expression (1) or (2) the number S. We also say that series (3) *converges to* S.

Remark. The equality $\lim\limits_{n\to\infty} S_n = S$, where S_n and S are complex numbers is defined in the same way as for real S_n, S, i.e. it means that $\forall\varepsilon > 0$, $\exists N$: $|S_n - S| < \varepsilon$, $\forall n > N$. Here $|S_n - S|$ is the modulus of the difference of two complex numbers S_n and S. For complex variables it is proved just in the same manner as for real variables that the limit of a sum, difference, product, and the quotient of the variables u_n, v_n is respectively equal to the sum, difference, product, and the quotient of the limits of these variables with the usual stipulation for the case of quotient ($\lim v_n \neq 0$).

By Cauchy's criterion (which holds for sequences of complex numbers as well), *for series* (1) *to be convergent, it is necessary and sufficient that, given any* $\varepsilon > 0$, *there should be N such that for all natural n > N and any natural p the following inequality holds*:

$$|u_{n+1} + \ldots + u_{n+p}| = |S_{n+p} - S_n| < \varepsilon.$$

Hence, in particular, follows ($p = 1$) that if series (1) is convergent, then its general term tends to zero:

$$\lim_{n\to\infty} u_n = 0. \tag{4}$$

Condition (4) is necessary for series (1) to be convergent, but, as will be seen from examples, is not sufficient.

Let us also consider the series

$$u_{n+1} + u_{n+2} + \ldots = \sum_{k=1}^{\infty} u_{n+k}. \tag{5}$$

Since conditions of Cauchy's criterion are stated in exactly the same manner for both series (1) and (5), they are simultaneously convergent or divergent. If they are convergent, then the sum of series (5) is equal to

$$\lim_{m\to\infty} \sum_{k=1}^{m} u_{n+k} = \lim_{m\to\infty} (S_{n+m} - S_n) = S - S_n.$$

Series (5) is called the *remainder* or the *remainder term of series* (1).

If the terms of series (1) are real and nonnegative, then its partial sums form a nondecreasing sequence $S_1 \leqslant$

$\leqslant S_2 \leqslant S_3 \leqslant \ldots$, and therefore if this sequence is bounded, i.e. if

$$S_n \leqslant M \quad (n = 1, 2, \ldots),$$

then the series is convergent and its sum satisfies the inequality

$$\lim_{n\to\infty} S_n = S \leqslant M.$$

If the sequence is unbounded, then the series is divergent:

$$\lim_{n\to\infty} S_n = \infty.$$

In this case we write

$$\sum_{k=0}^{\infty} u_k = \infty.$$

Example. The nth partial sum of the series

$$1 + z + z^2 + \ldots \tag{6}$$

is $S_n(z) = (1 - z^{n+1})/(1 - z)$ (for $z \neq 1$). If $|z| < 1$, then $|z^{n+1}| = |z|^{n+1}$ and $z^{n+1} \to 0$ $(n \to \infty)$. Hence, series (6) is convergent and has a sum equal to $(1 - z)^{-1}$ in the open circle $|z| < 1$. And if $|z| \geqslant 1$, then series (6) is divergent because in this case its general term, having the modulus not less than unity ($|z^n| \geqslant 1$), does not tend to zero as $n \to \infty$.

Sec. 9.2. Improper Integral and Series

Let us consider the integral

$$\int_a^b f(x)\,dx, \tag{1}$$

with only one singularity at the point b. Let

$$a = b_0 < b_1 < b_2 < \ldots < b, \; b_k \to b,$$

Then we can determine the series

$$\int_{b_0}^{b_1} f(x)\,dx + \int_{b_1}^{b_2} f(x)\,dx + \ldots = \sum_{k=0}^{\infty} \int_{b_k}^{b_{k+1}} f\,dx, \tag{2}$$

whose kth term is equal to $u_k = \int\limits_{b_k}^{b_{k+1}} f\,dx$.

Theorem 1. *If integral* (1) *is convergent, then series* (2) *is also convergent and there takes place the equality*

$$\int_a^b f(x)\,dx = \sum_0^{\infty} \int_{b_k}^{b_{k+1}} f\,dx. \tag{3}$$

Indeed,

$$\lim_{n\to\infty} \sum_0^n \int_{b_k}^{b_{k+1}} f\,dx = \lim_{n\to\infty} \int_{b_0}^{b_{n+1}} f\,dx = \int_a^b f\,dx.$$

If f is nonnegative on $[a, b)$, then, conversely, the convergence of series (2) implies the convergence of integral (1). Indeed, let the series be convergent and have the sum equal to S. For any b', where $a < b' < b$, there is $n' = n(b')$ such that $\forall n > n'$, $b' < b_n$. Therefore, taking into account that $f(x) \geqslant 0$,

$$\int_a^{b'} f\,dx \leqslant \int_a^{b_n} f\,dx = \sum_{k=0}^{n-1} \int_{b_k}^{b_{k+1}} f\,dx \leqslant S,$$

i.e. the integral in the left-hand side is bounded and, consequently, improper integral (1) does exist. But then, as it was proved above, equality (3) is valid.

And if the function f does not retain sign on $[a, b)$, then, generally speaking, the convergence of series (2) does not imply the convergence of the integral.

For instance, the series

$$\sum_{k=0}^{\infty} \int_{2k\pi}^{2(k+1)\pi} \sin x\,dx = \sum_0^{\infty} 0 = 0$$

is convergent, and the integral $\int_0^\infty \sin x\,dx$ is divergent, since the function of x

$$\int_0^x \sin t\,dt = 1 - \cos x$$

does not tend to the limit as $x \to \infty$.

Theorem 2. *If a function $f(x) \geqslant 0$ is continuous and does not increase on $[0, \infty)$, then both the integral*

$$\int_0^\infty f(x)\,dx \tag{3'}$$

and the series

$$\sum_{k=0}^{\infty} f(k) = f(0) + f(1) + f(2) + \dots \tag{4}$$

are simultaneously convergent or divergent.

Proof. There take place the inequalities

$$f(k+1) \leqslant \int_k^{k+1} f(x)\,dx \leqslant f(k) \quad (k = 0, 1, \dots).$$

Summing them with respect to k we obtain

$$\sum_1^{n+1} f(k) = \sum_0^n f(k+1) \leqslant \int_0^{n+1} f(x)\,dx \leqslant \sum_0^n f(k).$$

Hence, taking into consideration that all the terms in these relations are not monotonically decreasing with an increase in n, there follows the statement of the theorem.

It follows from this theorem that the series

$$1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \dots \tag{5}$$

converges for $\alpha > 1$ and diverges for $\alpha \leqslant 1$, since the function $1/(1 + x)^\alpha$ is continuous for $\alpha > 0$ and monoto-

nically decreases on $[0, \infty)$, and

$$\int_0^\infty \frac{dx}{(1+x)^\alpha} \begin{cases} <\infty & (\alpha > 1), \\ =\infty & (\alpha \leqslant 1). \end{cases}$$

Series (5) for $0 < \alpha \leqslant 1$ is an example of a divergent series with the general term ($u_n = n^{-\alpha}$) tending to zero.

In the case $\alpha \leqslant 0$ it is directly seen that series (5) is divergent (the general term does not tend to zero).

Sec. 9.3. Operations with Series

If $\sum_0^\infty u_k$ *and* $\sum_0^\infty v_k$ *are convergent series and* α *is a number, then the series* $\sum_0^\infty \alpha u_k$, $\sum_0^\infty (u_k \pm v_k)$ *are also convergent and we have*

$$\sum_0^\infty \alpha u_k = \alpha \sum_0^\infty u_k, \tag{1}$$

$$\sum_0^\infty (u_k \pm v_k) = \sum_0^\infty u_k \pm \sum_0^\infty v_k. \tag{2}$$

Indeed,

$$\sum_0^\infty \alpha u_k = \lim_{n\to\infty} \sum_0^n \alpha u_k = \alpha \lim_{n\to\infty} \sum_0^n u_k = \alpha \sum_0^\infty u_k,$$

$$\sum_0^\infty (u_k \pm v_k) = \lim_{n\to\infty} \sum_0^n (u_k \pm v_k)$$

$$= \lim_{n\to\infty} \sum_0^n u_k \pm \lim_{n\to\infty} \sum_0^n v_k = \sum_0^\infty u_k \pm \sum_0^\infty v_k.$$

It should be stressed that, generally speaking, the convergence of series on the left-hand side of (2) does not imply the convergence of either of the series on the right-hand side of (2), which is readily confirmed by the example of the series

$$(1 - 1) + (1 - 1) + \ldots . \tag{3}$$

This series is convergent, since all its terms are equal to zero, but the expression $\sum_0^\infty 1 - \sum_0^\infty 1$ does not make sense because the series entering into it are divergent.

If a series

$$u_0 + u_1 + u_2 + \ldots \tag{4}$$

is convergent and has a sum S, then its terms can be arbitrarily grouped in parentheses (without rearranging them!), for instance,

$$u_0 + (u_1 + u_2) + (u_3 + u_4 + u_5) + \ldots .$$

The new series thus formed whose terms are the sums of the numbers in parentheses is also convergent and its sum is again the number S, for the partial sums of the new series form a subsequence of the convergent sequence of partial sums of the original series (4).

On the contrary, if we are given a series whose terms are certain sums enclosed in parentheses, then, generally speaking, it is illegitimate to remove the parentheses, since this can destroy the convergence of the original series and can change its sum. For instance, series (3) is convergent, but if the parentheses are removed, then we obtain the divergent series $1 - 1 + 1 - \ldots$. However, if the terms in the parentheses are only nonnegative or nonpositive numbers, then the removal of the parentheses in such a series does not, of course, affect its convergence and its sum.

Sec. 9.4. Series with Nonnegative Terms

Theorem 1 (Comparison Tests for Positive Series). *Let there be given two series*

$$(1)\quad \sum_0^\infty u_k, \quad \textit{and} \quad (2)\quad \sum_0^\infty v_k$$

with nonnegative terms.

(a) *If* $u_k \leqslant v_k$ $(k = 0, 1, \ldots)$, *then the convergence of series* (2) *implies the convergence of series* (1) *and the divergence of series* (1) *implies the divergence of series* (2).

(b) *If*

$$\lim_{k\to\infty} \frac{u_k}{v_k} = A > 0, \qquad (1)$$

then series (1) *and* (2) *are simultaneously convergent or divergent.*

Proof. Let series (2) be convergent and let S be its sum. Then

$$\sum_0^n u_k \leqslant \sum_0^n v_k \leqslant S \quad (n = 0, 1, \ldots),$$

which means that the partial sums of series (1) are bounded and, hence, series (1) is convergent. Its sum S' satisfies the inequality $S' \leqslant S$.

Now suppose that series (1) is divergent; then (see Sec. 9.1) its nth partial sum increases indefinitely together with n, whence, by virtue of the inequality

$$\sum_0^n u_k \leqslant \sum_0^n v_k \quad (n = 0, 1, \ldots),$$

it follows that the nth partial sum of series (2) also increases indefinitely, that is, the latter series is divergent. This proves (a).

Let now the equality (1) take place. We take a positive number ε satisfying the inequality $A - \varepsilon > 0$. From (1) there follow the inequalities

$$A - \varepsilon < \frac{u_k}{v_k} < A + \varepsilon, \quad k > N,$$

which are true for a sufficiently large N, or the inequalities

$$(A - \varepsilon) v_k < u_k < (A + \varepsilon) v_k, \quad k > N. \qquad (2$$

If series (2) is convergent, then the series $\sum\limits_{k=N+1}^{\infty} (A + \varepsilon) v_k$ is also convergent and, by virtue of the second inequality of (2), the series $\sum\limits_{k=N+1}^{\infty} u_k$ is convergent together with the former. Therefore, series (1) is convergent. Conversely, the convergence of series (1) implies

the convergence of the series $\sum\limits_{k=N+1}^{\infty}(A-\varepsilon)v_k$ and, hence, the convergence of series (2).

We can prove in a similar manner that the divergence of one series implies the divergence of the other. This proves (b). Thus, the theorem has been proved.

Theorem 2 (D'Alembert's* Tests). *Let there be given a series*

$$\sum_{0}^{\infty} u_k \tag{3}$$

with positive terms.

(a) *If*

$$\frac{u_{k+1}}{u_k} \leqslant q < 1 \quad (k=0,\ 1,\ 2,\ \ldots), \tag{4}$$

then series (3) *is convergent; and if*

$$\frac{u_{k+1}}{u_k} \geqslant 1 \quad (k=0,\ 1,\ 2,\ \ldots), \tag{5}$$

series (3) *is divergent.*

(b) *If*

$$\lim_{k\to\infty} \frac{u_{k+1}}{u_k} = q, \tag{6}$$

then series (3) *is convergent for* $q<1$ *and divergent for* $q>1$.

Proof. We have

$$u_n = u_0 \frac{u_1}{u_0}\cdot\frac{u_2}{u_1}\cdots\frac{u_n}{u_{n-1}} \quad (n=0,\ 1,\ 2,\ \ldots), \tag{7}$$

therefore from (4) it follows that

$$u_n \leqslant u_0 q^n, \quad q<1 \quad (n=0,\ 1,\ \ldots),$$

and, since the series $\sum\limits_{1}^{\infty} u_0 q^n$ is convergent, series (3) is also convergent. It follows from (5) that $u_n \geqslant u_0$ ($n=0$, 1, 2, . .) and, since $u_0>0$, series (3) is divergent (the general term does not tend to zero).

---

* D'Alembert, Jean Le Rond (1717-1783). French mathematician, philosopher, and physicist.

Now, if property (6) holds and $q < 1$, then for any positive ε satisfying the condition $q + \varepsilon < 1$ we have $u_{k+1}/u_k < q + \varepsilon < 1$ $(k \geqslant N)$, where N is sufficiently large. By test (4), in this case the series $\sum\limits_{N}^{\infty} u_k$ is convergent and, together with it, series (3) is also convergent.

If property (6) is fulfilled for $q > 1$, then it follows that $u_{k+1}/u_k > 1$ $(k \geqslant N)$ for a sufficiently large N and, therefore, by test (5), the series $\sum\limits_{N}^{\infty} u_k$ is divergent and, together with it, so is series (3).

Theorem 3 (Cauchy's Test). *Let there be given series* (3) *with positive terms.*

(a) *If*

$$\sqrt[k]{u_k} < q < 1 \quad (k = 0,\ 1,\ \ldots), \tag{8}$$

then series (3) *is convergent; and if*

$$\sqrt[k]{u_k} \geqslant 1 \quad (k = 0,\ 1,\ \ldots), \tag{9}$$

then series (3) *is divergent.*

(b) *If*

$$\lim_{k \to \infty} \sqrt[k]{u_k} = q, \tag{10}$$

then series (3) *is convergent for* $q < 1$ *and divergent for* $q > 1$.

(c) *If*

$$\overline{\lim} \sqrt[k]{u_k} = q, \tag{11}$$

then series (3) *is convergent for* $q < 1$ *and divergent for* $q > 1$.

Proof. It follows from inequality (8) that $u_k < q^k$ $(q < 1,\ k = 0,\ 1,\ \ldots)$, and, since in this case the series $\sum\limits_{0}^{\infty} q^k$ is convergent, so is series (3). It follows from inequality (9) that $u_k \geqslant 1$ $(k = 0,\ 1,\ \ldots)$, i.e. the necessary condition for convergence is not fulfilled, and therefore series (3) is divergent.

Further, from property (10) with $q < 1$ it follows tha

$$\sqrt[k]{u_k} < q + \varepsilon < 1 \quad (k \geqslant N) \tag{12}$$

for sufficiently large N, whence

$$u_k < (q + \varepsilon)^k \quad (k \geqslant N),$$

and, since the series $\sum\limits_N^\infty (q + \varepsilon)^k$ is convergent, the last inequality implies that $\sum\limits_N^\infty u_k$ is convergent and, hence, series (3) is also convergent. If property (10) is fulfilled for $q > 1$, then it follows that $\sqrt[k]{u_k} > 1$, i.e. $u_k > > 1$ $(k \geqslant N)$ for sufficiently large N, whence obviously follows the divergence of series (3).

From property (11) (as well as from property (10)) with $q < 1$ there follows (12), whence, as has been proved, there follows the convergence of series (3).

Finally, let property (11) be fulfilled for $q > 1$. Let us choose a finite number q_1 so that $1 < q_1 < q$. On the basis of the property of the upper limit (see Sec. 2.10), there exists a subsequence $k_1 < k_2 < \ldots$ such that

$$\sqrt[k_s]{u_{k_s}} > q_1 > 1 \quad (s = 1, 2, \ldots),$$

i.e.

$$u_{k_s} > q_1^{k_s}.$$

But then $\lim\limits_{s \to \infty} u_{k_s} = \infty$ and series (3) diverges.

Remark. The series with the general term $u_n = = n^{-\alpha}$ $(\alpha > 0)$ is convergent when $\alpha > 1$ and divergent when $\alpha \leqslant 1$ (see Sec. 9.2, (5)). In both cases we have

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = 1, \tag{13}$$

and also

$$\lim_{n \to \infty} \sqrt[n]{u_n} = 1. \tag{14}$$

This shows that there are both convergent and divergent series for which conditions (13) and (14) are fulfilled.

The series $1 + 1/2 + 1/3 + \ldots$ is called the *harmonic series*. It is divergent (see Sec. 9.2, (5)).

Examples.

$$(1)\ \sum_{0}^{\infty} \frac{x^k}{k!}. \quad (2)\ \sum_{1}^{\infty} \frac{x^k}{k^\alpha}\ (\alpha > 0). \quad (3)\ \sum_{1}^{\infty} (e^{1/k} - 1).$$

$$(4)\ \sum_{1}^{\infty} \ln\left(1 + \frac{1}{k}\right). \quad (5)\ \sum_{1}^{\infty} q^{k+\sqrt{k}} \quad (q > 0).$$

Series (1) is convergent $\forall x \geqslant 0$. For $x = 0$ it is obvious, and for $x > 0$ it follows from the fact that $u_{k+1}/u_k = x/(k+1) \to 0$, $k \to \infty$. Moreover, we know that this series is Taylor's series of the function e^x and converges for any x to the sum equal to e^x.

As to series (2), it is convergent for $0 \leqslant x < 1$ and divergent for $x > 1$, since for $x > 0$ for this series we have $u_{k+1}/u_k = x\,(k/(k+1))^\alpha \to x$, $k \to \infty$; for the case $x = 1$ see the above remark. The case $x = 0$ is trivial.

Series (3) and (4) are divergent by virtue of Theorem 1 proved at the beginning of this section, since $e^{1/k} - 1 \approx 1/k$ $(k \to \infty)$ and $\ln(1 + (1/k)) \approx 1/k$ $(k \to \infty)$ (here "$\approx$" is the sign of asymptotic equality, see Secs. 3.9 and 3.10), and the harmonic series $\sum_{1}^{\infty} k^{-1}$ is divergent.

Series (5) is convergent for $0 \leqslant q < 1$ and divergent for $q > 1$, because for this series we have $\sqrt[k]{u_k} = q^{1+(1/\sqrt{k})} \to q$ $(k \to \infty)$. It is also divergent for $q = 1$, since in this case its general term is equal to 1.

Theorem 4. *Let a series*

$$u_0 + u_1 + u_2 + \ldots \tag{15}$$

with nonnegative terms be convergent to a sum S. Then the series

$$u_0' + u_1' + u_2' + \ldots \tag{16}$$

obtained from the former by rearranging and renumbering its terms in an arbitrary way is also convergent and has the same sum S.

Proof. Let

$$S'_n = u'_0 + u'_1 + \ldots + u'_n$$

be the nth partial sum of series (16). Its terms have some indices $k_0, k_1, \ldots, k_n$ as terms of the original series (15). Let N be the greatest among them and S_N be the Nth partial sum of series (15). We, obviously, have $S'_n \leqslant S_N \leqslant S$, and, since n is arbitrary, series (16) is convergent and has a sum $S' \leqslant S$. Now interchanging the roles of series (15) and (16), we can repeat the same argument, whence $S \leqslant S'$. Therefore $S = S'$.

Sec. 9.5. Leibniz' Series

A series of the form

$$a_0 - a_1 + a_2 - a_3 + \ldots, \tag{1}$$

where the numbers $a_k > 0$ form a monotone decreasing sequence convergent to zero ($a_k \geqslant a_{k+1}$; $a_k \to 0$, $k \to \infty$), is called a *Leibniz series.*

Let us show that a Leibniz series is convergent and that its sum satisfies the inequality $S \leqslant a_0$.

Indeed, its partial sum S_{2n+1} with an odd index $2n + 1$ can be written in the form

$$S_{2n+1} = a_0 - (a_1 - a_2) - (a_3 - a_4) - \ldots$$
$$\ldots - (a_{2n-1} - a_{2n}) - a_{2n+1},$$

whence it, obviously, follows that it is bounded above by the number a_0:

$$S_{2n+1} \leqslant a_0.$$

On the other hand, this sum can be written in the form

$$S_{2n+1} = (a_0 - a_1) + (a_2 - a_3) + \ldots + (a_{2n} - a_{2n+1}),$$

whence it follows that it is monotone nondecreasing. Therefore there exists the limit $\lim\limits_{n\to\infty} S_{2n+1} = S \leqslant a_0$. It is also obvious that

$$\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} (S_{2n+1} - a_{2n+1}) = S - 0 = S.$$

The theorem has been proved.

Example. The series $1-\frac{1}{2}+\frac{1}{3}-\dots$ is obviously a Leibniz series; hence, it is convergent and its sum S does not exceed 1 (as follows from Sec. 4.16, its sum S is in fact equal to $\ln 2$).

Sec. 9.6. Absolutely Convergent Series

A series

$$u_0+u_1+u_2+\dots \tag{1}$$

with complex terms is said to be *absolutely convergent* if the series

$$|u_0|+|u_1|+|u_2|+\dots \tag{2}$$

of the moduli of its terms is convergent.

An absolutely convergent series is always convergent.

Indeed, suppose that series (1) is absolutely convergent; then series (2) is convergent and, by Cauchy's criterion, given any $\varepsilon>0$, there is N such that $\varepsilon>|u_{n+1}|+ +\dots+|u_{n+p}|$ for all p and $n>N$. Then moreover, $\varepsilon>|u_{n+1}+\dots+u_{n+p}|$ and, by Cauchy's criterion, series (1) is convergent.

Convergent series with nonnegative terms converge absolutely in a trivial way. The series $1-\frac{1}{2^\alpha}+\frac{1}{3^\alpha}-\dots$ $\dots$ $(\alpha>0)$ is convergent, since it is a Leibniz series, but it is absolutely convergent only for $\alpha>1$.

Theorem. *If a series is absolutely convergent, then any rearrangement of its terms does not violate its absolute convergence, and the sum of the new series remains unchanged.*

Proof. We shall begin with the special case when the terms u_k of the series are real numbers.

Let us set (for real u_k)

$$u_k^+=\begin{cases} u_k, & \text{if } u_k\geqslant 0,\\ 0, & \text{if } u_k<0,\end{cases}\qquad u_k^-=\begin{cases} -u_k, & \text{if } u_k\leqslant 0,\\ 0, & \text{if } u_k>0.\end{cases} \tag{3}$$

The numbers u_k^+ and u_k^- are, obviously, nonnegative an

$$u_k=u_k^+-u_k^-. \tag{4}$$

Together with series (1), we shall consider the two series

$$\sum_{0}^{\infty} u_k^+ \text{ and } \sum_{0}^{\infty} u_k^- \tag{5}$$

(with nonnegative terms).

Let series (1) be absolutely convergent and let its terms be real numbers u_k. Then series (5) are also convergent, since we, obviously, have $u_k^+ \leqslant |u_k|$, $u_k^- \leqslant |u_k|$.

Let the series obtained by rearranging the terms of the original series (1) have the form $v_1 + v_2 + v_3 + \ldots$. For its terms we can introduce the corresponding numbers v_k^+ and v_k^- as was done for series (1). Then (see the explanations below)

$$\sum_{0}^{\infty} u_k = \sum_{0}^{\infty} (u_k^+ - u_k^-) = \sum_{0}^{\infty} u_k^+ - \sum_{0}^{\infty} u_k^-$$

$$= \sum_{0}^{\infty} v_k^+ - \sum_{0}^{\infty} v_k^- = \sum_{0}^{\infty} (v_k^+ - v_k^-) = \sum_{0}^{\infty} v_k.$$

The first equality in this chain follows from (4), the second follows from (2) of Sec. 9.3 if we take into consideration that series (5) are convergent; the third equality follows from the fact that any convergent series with nonnegative terms is rearrangeable, the fourth equality is implied by (2) from Sec. 9.3, and, finally, the fifth equality is fulfilled because $v_k = v_k^+ - v_k^-$. Thus, for real u_k the theorem has been proved.

Now we pass to the general case when $u_k = \alpha_k + i\beta_k$ are complex numbers and, as before, v_k denote the terms of the rearranged series. Since $|\alpha_k| \leqslant |u_k|$ and $|\beta_k| \leqslant \leqslant |u_k|$, the series $\sum_{0}^{\infty} \alpha_k$ and $\sum_{0}^{\infty} \beta_k$ (with real terms) are absolutely convergent and their terms, as was proved above, can be rearranged arbitrarily; therefore, setting $v_k = \gamma_k + i\delta_k$, we obtain

$$\sum_{0}^{\infty} u_k = \sum_{0}^{\infty} (\alpha_k + i\beta_k) = \sum_{0}^{\infty} \alpha_k + i \sum_{0}^{\infty} \beta_k$$

$$= \sum_{0}^{\infty} \gamma_k + i \sum_{0}^{\infty} \delta_k = \sum_{0}^{\infty} (\gamma_k + i\delta_k) = \sum_{0}^{\infty} v_k.$$

The proof of the theorem has been completed.

Sec. 9.7. Conditionally and Unconditionally Convergent Series with Real Terms

We know from the preceding section that an absolutely convergent series with real or complex terms remains after any rearrangement of its terms absolutely convergent and its sum remains the same.

It turns out that this property, i.e. to retain the sum despite any rearrangement of terms is inherent only in absolutely convergent series.

Let us consider a series

$$u_0 + u_1 + u_2 + \ldots \tag{1}$$

with real terms which is convergent, but not absolutely.

We can prove that for any number S, finite or infinite, i.e. satisfying the inequalities $-\infty \leqslant S \leqslant +\infty$, there is a rearrangement of the terms of series (1) which result in a series convergent to S.

We shall call a series *unconditionally convergent* if any rearrangement of its terms does not violate its convergence and if it converges to one and the same sum S. A series is said to be *conditionally convergent* if it is convergent and if its terms can be rearranged so that the convergence is violated or if it remains convergent, then it converges to another sum.

It follows from the aforesaid that *for a series to be absolutely convergent, it is necessary and sufficient that it be unconditionally convergent.*

We should like to close this section with the following remark. Let there be given series (1) constructed from real numbers which is convergent, but not absolutely. Series (1) contains an infinitude of positive and negative terms and, obviously, when taken separately, they form divergent series (otherwise the initial series would be absolutely convergent).

Sec. 9.8. Sequences and Series of Functions. Uniform Convergence

Let us consider a sequence of functions $\{f_k(x)\}$ defined on a set of points $x = (x_1, \ldots, x_n)$ of the n-dimensional space. They may take on complex values ($f_k(x) =$

$= \alpha_k(x) + i\beta_k(x))$. Let us also assume that x is a complex point ($x = \xi + i\eta$), running through a set E in a complex plane; then $f_k(x)$ are functions of the complex variable x.

Let the sequence $\{f_n(x)\}$ tend to a number $f(x)$ for every fixed $x \in E$.

A sequence $f_n(x)$ is said to be *uniformly convergent on E to $f(x)$* if there exists a sequence of nonnegative numbers ρ_n (independent of x) such that

$$|f(x) - f_n(x)| \leqslant \rho_n, \quad \forall x \in E. \tag{1}$$

This definition is equivalent to the following: for any $\varepsilon > 0$ there is N such that for $n > N$

$$|f(x) - f_n(x)| < \varepsilon, \quad \forall x \in E.$$

If the conditions of the former definition are fulfilled, then for any $\varepsilon > 0$ there is N such that

$$\varepsilon \underset{n>N}{>} \rho_n \geqslant |f(x) - f_n(x)|, \quad \forall x \in E,$$

i.e.

$$\varepsilon > |f(x) - f_n(x)|, \quad \forall x \in E, \quad n > N. \tag{2}$$

Conversely, if the conditions of the latter definition hold, then for any $\varepsilon > 0$ there exists N such that inequality (2) is fulfilled. But then

$$\varepsilon \geqslant \sup_{x \in E} |f(x) - f_n(x)| = \rho_n, \quad n > N. \tag{3}$$

We see that the nonnegative numbers ρ_n are independent of x and $|f(x) - f_n(x)| \leqslant \rho_n$, $\rho_n \to 0$, i.e. the conditions of the first definition are fulfilled.

In the first definition $\sup\limits_{x \in E} |f(x) - f_n(x)| = \rho_n$ can be taken as ρ_n. If it tends to zero as $n \to \infty$ ($\rho_n \to 0$), then $f_n(x)$ tends to $f(x)$ uniformly on E, if it does not, then nonuniformly.

We can also give a third definition of uniform convergence (in terms of Cauchy's criterion): a sequence $\{f_n(x)\}$ is uniformly convergent on E if, given any $\varepsilon > 0$, there is N such that the inequality

$$|f_{n+p}(x) - f_n(x)| < \varepsilon \tag{4}$$

is fulfilled for any $n > N$ and $p > 0$ and for all $x \in E$.

If a sequence is uniformly convergent in the sense of the second definition, then it follows that for any $\varepsilon > 0$ there is N such that for $n > N$ and any p there holds the inequality

$$| f_{n+p}(x) - f_n(x) | \leqslant | f_{n+p}(x) - f(x) | + | f(x) - f_n(x) | < 2\varepsilon, \quad \forall x \in E,$$

i.e. the conditions of the third definition are fulfilled. On the other hand, if the conditions of the third definition hold, then for every fixed $x \in E$ the ordinary Cauchy's criterion for number sequences holds, and therefore the sequence is convergent on E to a function $f(x)$. Now, taking an arbitrary $\varepsilon > 0$, we can find the number N as indicated in the third definition and pass to the limit as $p \to \infty$ in inequality (4), where $n > N$ is fixed; this results in $| f(x) - f_n(x) | \leqslant \varepsilon$ $(x \in E)$, whence

$$\rho_n = \sup_{x \in E} | f(x) - f_n(x) | \leqslant \varepsilon,$$

and, since $n > N$ can be taken arbitrarily, we have $\rho_n \to 0$ $(n \to \infty)$, i.e. the conditions of the first definition are fulfilled.

Let us represent in a rectangular coordinate system the graph of the function $y = f(x)$ (a limit function) which is regarded as continuous on $[a, b]$ (Fig. 105). Let us take $\varepsilon > 0$ and determine the ε-strip 2ε wide surrounding the graph. An arbitrary point of the ε-strip with abscissa $x \in [a, b]$ has an ordinate y satisfying the inequalities

$$f(x) - \varepsilon < y < f(x) + \varepsilon.$$

If the sequence of functions $\{f_n(x)\}$ tends to $f(x)$ uniformly on $[a, b]$, then, given $\varepsilon > 0$, we can indicate an N such that for any $n > N$ the graph $y = f_n(x)$ will be found inside the ε-strip. And if $f_n(x)$ tends to $f(x)$ non-uniformly on $[a, b]$, then, although for every value of x $f_n(x)$ tends to $f(x)$, nevertheless, for any $\varepsilon > 0$ it is impossible to indicate an N such that for every $n > N$ all the graphs $y = f_n(x)$ be contained in the ε-strip (see Example 3 below).

It is easy to see that if α is a number and $\{f_k(x)\}$ and $\{\varphi_k(x)\}$ are two sequences of functions uniformly convergent on E, then the sequences $\{\alpha f_k(x)\}$ and $\{f_k(x) \pm \pm \varphi_k(x)\}$ are also uniformly convergent on E. It can also be readily shown that if a sequence of functions is uniformly convergent on E, then it is also uniformly convergent on every subset $E' \subset E$. The converse is, generally speaking, not true.

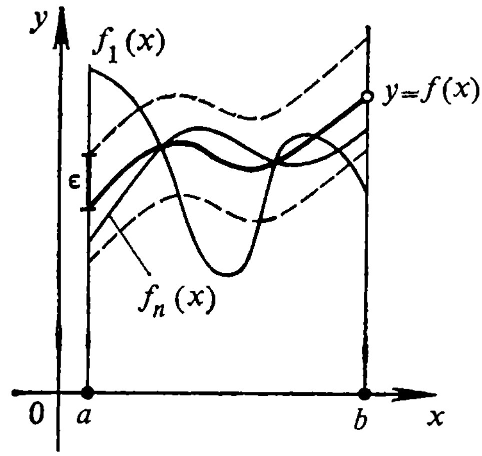

Fig. 105

Note that to every sequence of functions $\{f_k(x)\}$ there corresponds the series

$$f_0(x) + [f_1(x) - f_0(x)] + [f_2(x) - f_1(x)] + \ldots,$$

whose nth partial sums are respectively equal to $f_n(x)$.

Now let us consider a series

$$u_0(x) + u_1(x) + u_2(x) + \ldots, \tag{5}$$

whose terms are, generally speaking, complex functions of $x \in E$, where as before, E is a point set in the n-dimensional space or in the complex plane.

Series (5) *is uniformly convergent on the set E to a function $S(x)$* if the sequence of partial sums $\{S_k(x)\}$ is uniformly convergent to $S(x)$ on E.

In particular, the definition of a uniformly convergent series can be stated as follows: series (5) is uniformly convergent on the set E if for any $\varepsilon > 0$ there is N such that for $n > N$ and $p > 0$ and for any $x \in E$ there holds the inequality

$$|u_{n+1}(x) + \ldots + u_{n+p}(x)| < \varepsilon.$$

The theorem below provides an important test for uniform convergence of a series.

Theorem 1 (of Weierstrass). *If the terms of series* (5) *satisfy the inequalities*

$$|u_k(x)| \leqslant \alpha_k \quad (k = 0, 1, \ldots), \tag{6}$$

where $x \in E$ *and* α_k *are numbers (independent of* x*) such that the series with the terms* α_k *is convergent, then series* (5) *is uniformly and absolutely convergent on* E.

Indeed, if the series with the terms α_k is convergent, then it follows from (6) that for any $\varepsilon > 0$ there is N such that for any $n > N$ and $p > 0$ and for arbitrary $x \in E$ the inequality

$$\varepsilon > \alpha_{n+1} + \ldots + \alpha_{n+p} \geqslant | u_{n+1}(x) | + \ldots$$
$$\ldots + | u_{n+p}(x) | \geqslant | u_{n+1}(x) + \ldots + u_{n+p}(x) |$$

is fulfilled. This means that series (5) is uniformly convergent on E. Its absolute convergence is evident.

Theorem 2. *If a sequence of functions* $\{f_n(x)\}$ *is uniformly convergent on a set* E *to a function* f, *and* f_n *are continuous (relative to* E*) at a point* x^0, *then the function* f *is also continuous at* x^0.

This theorem can be restated in terms of series: *the sum of a uniformly convergent series of functions continuous at a point* $x^0 \in E$ *is a continuous function at that point**.

Proof. Let us take $\varepsilon > 0$ and choose a natural N such that $| f(x) - f_N(x) | < \varepsilon/3$ for all $x \in E$. This is possible, since the sequence of functions f_n is uniformly convergent to f on E. We have further

$$|f(x)-f(x^0)| \leqslant |f(x)-f_N(x)| + |f_N(x)-f_N(x^0)|$$
$$+ |f_N(x^0)-f(x^0)| < 2\frac{\varepsilon}{3} + |f_N(x)-f_N(x^0)| \qquad (7)$$

for any point $x \in E$. But the function f_N is continuous at x^0 and there is $\delta > 0$ such that $| f_N(x) - f_N(x^0) | < \varepsilon/3$ for all $x \in E$ such that $| x - x^0 | < \delta$; therefore it follows from (7) that for such x's

$$|f(x)-f(x^0)| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

and the theorem has been proved.

Example 1. The series

$$1 + (x - 1) + (x^2 - x) + (x^3 - x^2) + \ldots \qquad (8)$$

is convergent on the closed interval $[0, 1]$, but not uniformly. On the closed interval $[0, q]$, where $0 < q < 1$, it is uniformly convergent.

* See the remark in Sec. 9.12.

Indeed, the nth partial sum of series (8)

$$S_n(x) = x^n \xrightarrow[n\to\infty]{} \begin{cases} 0, & 0 \leqslant x < 1, \\ 1 & x = 1. \end{cases}$$

The absolute value of the difference $S(x) - S_n(x)$ (the remainder) is equal to

$$|S(x) - S_n(x)| = \begin{cases} x^n, & 0 \leqslant x < 1, \\ 0, & x = 1. \end{cases} \qquad (9)$$

On the interval $[0, q]$, where $0 < q < 1$,

$$|S(x) - S_n(x)| = x^n \leqslant q^n.$$

The right-hand side of this inequality is independent of $x \in [0, q]$ and tends to zero as $n \to \infty$ ($q^n \to 0$). This shows that series (8) is uniformly convergent on the closed interval $[0, q]$, where $0 < q < 1$.

On the other hand, from equality (9) it is seen that

$$\sup_{x \in [0, 1]} |S(x) - S_n(x)| = 1.$$

Hence, the number 1 is the least number exceeding $|S(x) - S_n(x)|$ for all $x \in [0, 1]$. But a constant number 1 does not tend to zero as $n \to \infty$, therefore series (8) is although convergent on $[0, 1]$, but nonuniformly.

Example 2. The series

$$\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \ldots \qquad (10)$$

has the nth term satisfying the inequality

$$\frac{|\sin nx|}{n^2} \leqslant \frac{1}{n^2}, \quad \forall x \in (-\infty, \infty),$$

and the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

is convergent. Therefore, according to the theorem of Weierstrass, series (10) is uniformly convergent throughout the entire axis $(-\infty, \infty)$.

Since the terms of series (10) are continuous functions, by Theorem 2, the sum of this series is a continuous function.

Example 3. Figure 106 represents the function $f_n(x)$ ($n = 1, 2, \ldots$). It is linear on each of the subintervals $[0, 1/n]$, $[1/n, 2/n]$, $[2/n, 1]$, separately. Besides, $f_n(0) = f(2/n) = 0$ and $f_n(x) = 0$ on $[2/n, 1]$, $f_n(1/n) = 1$. Obviously,

$$\lim_{n\to\infty} f_n(x) = f(x) = 0,$$

$$\forall x \in [0, 1],$$

since $f_n(0) = 0 \to 0$, and if $0 < x \leqslant 1$, then $f_n(x) = 0$, $\forall n > 2/x$.

Further, it is obvious that

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} f_n(x) = 1,$$

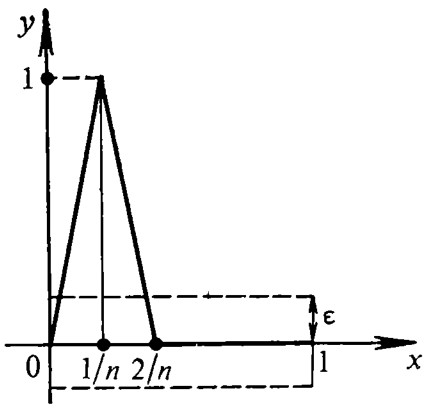

Fig. 106

and the constant number 1 does not tend to zero as $n \to \infty$, i.e. $f_n(x) \to f(x) \equiv 0$ on $[0, 1]$, but nonuniformly.

Shown in Fig. 106 by a dashed line is an ε-strip surrounding the limit curve $f(x) = 0$ $(0 \leqslant x \leqslant 1)$. For any n the graph of the function $f_n(x)$ is not contained entirely in the ε-strip. But this does not interfere with $f_n(x) \to f(x) = 0$, $\forall x \in [0, 1]$.

We shall also prove more subtle tests for uniform convergence based on the so-called *Abel identity* (which is an analogue of the formula for integration by parts).

Let us consider a series of the form

$$\alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots \tag{11}$$

where α_k and β_k are functions of $x \in E$ (or constant numbers). On making $B_k = \beta_{n+1} + \beta_{n+2} + \ldots + \beta_{n+k}$ we write for the truncated sum of series (11) Abel's identity

$$\sum_{k=1}^{p} \alpha_{n+k}\beta_{n+k} = \alpha_{n+1}\beta_{n+1} + \ldots + \alpha_{n+p}\beta_{n+p}$$

$$= \alpha_{n+1}B_1 + \alpha_{n+2}(B_2 - B_1) + \ldots + \alpha_{n+p}(B_p - B_{p-1})$$

$$= (\alpha_{n+1}-\alpha_{n+2}) B_1 + (\alpha_{n+2}-\alpha_{n+3}) B_2 + \ldots + (\alpha_{n+p-1}-\alpha_{n+p}) B_{p-1}$$

$$+\alpha_{n+p}B_p = \sum_{k=1}^{p-1} (\alpha_{n+k}-\alpha_{n+k+1}) B_k + \alpha_{n+p}B_p. \qquad (12)$$

Now we can easily establish the following two tests for uniform convergence (in the case of constant α_k and β_k, simply convergence) of series (11).

Theorem 3 (Dirichlet's Test for Uniform Convergence of a Series). *If the partial sums of the series*

$$\beta_0 + \beta_1 + \beta_2 + \ldots \qquad (13)$$

are uniformly bounded and the sequence of the real functions $\alpha_k(x)$ is uniformly (relative to x) convergent to zero (as k increases) on the set E and is decreasing for every fixed x, then series (11) *is uniformly convergent on E.*

Indeed, let M be a constant such that $|\sigma_n| < M$, where σ_n are the partial sums of series (13). Then for any n and k we have

$$|B_k| = |\sigma_{n+k} - \sigma_n| \leqslant |\sigma_{n+k}| + |\sigma_n| \leqslant 2M.$$

Therefore, by virtue of (12) and the fact that α_s is decreasing and uniformly tends to zero, we conclude that the inequality

$$\left|\sum_1^p \alpha_{n+k}\beta_{n+k}\right| \leqslant 2M \sum_{k=1}^{p-1} (\alpha_{n+k}-\alpha_{n+k+1}) + \alpha_{n+p}2M = 2M\,\alpha_{n+1} < \varepsilon$$

is fulfilled for any $n > N$ and p and any $x \in E$, provided that N is sufficiently large. Consequently, series (11) is uniformly convergent. The last inequality in this chain holds for all $x \in E$, since $\alpha_{n+1}(x)$ tends to zero uniformly as $n \to \infty$.

Theorem 4 (Abel's Test for Uniform Convergence of a Series). *If the real functions α_k form a monotone decreasing sequence (as k increases) for every fixed $x \in E$ and are uniformly bounded on E, and if series* (13) *is uniformly convergent on E, then series* (11) *is also uniformly convergent.*

Indeed, let $M \geqslant |\alpha_k|$ $(k = 0, 1, \ldots)$; the functions α_k can be positive (or negative!). By the uniform convergence of series (13), for any $\varepsilon > 0$ there is N such that $|B_k| < \varepsilon$ for any $n > N$ and k. Therefore, by virtue of (12) and by the monotonicity of α_s, for any $n > N$ and p we have

$$\left|\sum_1^p \alpha_{n+k}\beta_{n+k}\right| \leqslant \varepsilon \sum_{k=1}^{p-1} (\alpha_{n+k}-\alpha_{n+k+1}) + \varepsilon|\alpha_{n+p}|$$

$$= \varepsilon(\alpha_{n+1}-\alpha_{n+p}) + \varepsilon|\alpha_{n+p}| \leqslant 3\varepsilon M,$$

i.e. series (11) is uniformly convergent.

Example 4. The series

$$\sum_1^\infty \frac{\cos kx}{k^\alpha}, \quad \sum_1^\infty \frac{\sin kx}{k^\alpha} \; (\alpha > 0) \qquad (14)$$

are uniformly and absolutely convergent for $\alpha > 1$ throughout the real axis ($-\infty < x < \infty$), since the absolute values of their kth terms do not exceed $k^{-\alpha}$, and the series $\sum k^{-\alpha}$ is convergent for $\alpha > 1$, and, hence, by Weierstrass' test, the given series are uniformly convergent (for $\alpha > 1$). Weierstrass' test is no longer applicable when $\alpha \leqslant 1$, since in this case the series $\sum k^{-\alpha}$ is divergent. But if we take a narrower interval $[\varepsilon, 2\pi - \varepsilon]$, where ε is an arbitrary number such that $0 < \varepsilon < 2\pi - \varepsilon < 2\pi$ both series turn out to be uniformly convergent on $[\varepsilon, 2\pi - \varepsilon]$ for $0 < \alpha \leqslant 1$. Indeed, the partial sums of the series

$$\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \ldots, \qquad \sin x + \sin 2x + \ldots$$

are respectively equal to

$$D_n(x) = \frac{\sin\left(n + \frac{1}{2}\right)x}{2\sin\frac{x}{2}}, \quad K_n(x) = \frac{\cos\frac{x}{2} - \cos\left(n + \frac{1}{2}\right)x}{2\sin\frac{x}{2}} \tag{15}$$

$$(n = 1, 2, \ldots).$$

This can be readily checked by multiplying the partial sums of the series under consideration and dividing them by $2\sin(x/2)$ and by carrying out appropriate trigonometric transformation in the numerator. Functions (15) are uniformly bounded on $[\varepsilon, 2\pi - \varepsilon]$.

$$|D_n(x)| \leqslant \frac{1}{2\sin(\varepsilon/2)}, \quad |K_n(x)| \leqslant \frac{1}{\sin(\varepsilon/2)} \quad (n = 1, 2, \ldots);$$

besides $n^{-\alpha} \geqslant (n+1)^{-\alpha}$ and $n^{-\alpha} \to 0$, and therefore, by Dirichlet's test, series (14) are uniformly convergent on $[\varepsilon, 2\pi - \varepsilon]$.

Sec. 9.9. Integration and Differentiation of Uniformly Convergent Series

Theorem 1. *Let there be given a sequence $\{f_n(x)\}$ of (real or complex) continuous functions on a closed interval $[a, b]$ convergent to a function f. If the convergence is uniform, then*

$$\lim_{n\to\infty} \int_a^x f_n(t)\,dt = \int_a^x f(t)\,dt \tag{1}$$

uniformly on $[a, b]$. In particular (when $x = b$),

$$\lim_{n\to\infty} \int_a^b f_n(t)\,dt = \int_a^b f(t)\,dt. \tag{2}$$

Proof. From the conditions of the theorem it follows (see Sec. 9.8, Theorem 2) that the function f is continuous on $[a, b]$ and

$$\max_{a\leqslant t\leqslant b} |f_n(t) - f(t)| = r_n \to 0 \quad (n \to \infty).$$

Therefore

$$\left|\int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt\right| \leqslant \int_a^x |f_n(t) - f(t)|\,dt$$

$$\leqslant \int_a^b r_n\,dt = (b-a)\,r_n,$$

where the right-hand side is independent of x and tends to zero as $n \to \infty$, which proves the theorem.

Theorem 2. *A series*

$$S(x) = u_0(x) + u_1(x) + u_2(x) + \ldots \tag{3}$$

of (real or complex) continuous functions uniformly convergent on a closed interval $[a, b]$ *can be integrated termwise* ($a \leqslant x_0 \leqslant b$):

$$\int_{x_0}^x S(t)\,dt = \int_{x_0}^x u_0(t)\,dt + \int_{x_0}^x u_1(t)\,dt + \ldots\,. \tag{4}$$

The integrated series (4) *is uniformly convergent on* $[a, b]$.

In particular,

$$\int_a^b S(t)\,dt = \int_a^b u_0(t)\,dt + \int_a^b u_1(t)\,dt + \ldots\,. \tag{5}$$

Proof. Note that $S(x)$, as the sum of a series of continuous functions uniformly convergent on the closed interval $[a, b]$, is in its turn a continuous function on $[a, b]$. Let

$$S_n(x) = \sum_0^n u_k(x).$$

Since series (3) uniformly converges to $S(x)$, we have

$$\sup_{a\leqslant x\leqslant b} |S_n(x)-S(x)| = r_n \to 0 \quad (n\to\infty),$$

therefore

$$\left|\int_{x_0}^{x} S(t)\,dt - \sum_{0}^{n}\int_{x_0}^{x} u_k(t)\,dt\right| = \left|\int_{x_0}^{x} S(t)\,dt - \int_{x_0}^{x} S_n(t)\,dt\right|$$

$$= \left|\int_{x_0}^{x} [S(t)-S_n(t)]\,dt\right| \leqslant \int_{a}^{b} |S(t)-S_n(t)|\,dt$$

$$\leqslant (b-a)\,r_n \to 0 \quad (n\to\infty),$$

and the theorem has been proved.

Theorem 3. *Let*

$$u_0(x) + u_1(x) + u_2(x) + \ldots \tag{6}$$

be a series of (real or complex) continuously differentiable functions defined on a closed interval $[a, b]$. *If series* (6) *is convergent at a point* $x_0 \in [a, b]$ *and the series*

$$u_0'(x) + u_1'(x) + u_2'(x) + \ldots \tag{7}$$

obtained from (6) *by the formal term-by-term differentiation is uniformly convergent on* $[a, b]$, *then series* (6) *is uniformly convergent on* $[a, b]$, *its sum* $S(x)$ *is a differentiable function, and the derivative of* $S(x)$ *is equal to the sum of series* (7).

Thus

$$S(x) = u_0(x) + u_1(x) + u_2(x) + \ldots, \tag{8}$$

$$S'(x) = u_0'(x) + u_1'(x) + u_2'(x) + \ldots \qquad (a \leqslant x \leqslant b). \tag{9}$$

Proof. By the hypothesis, series (7) is uniformly convergent on $[a, b]$ and its terms are continuous functions on $[a, b]$. Therefore its sum, denoted for the time being by $\varphi(x)$, is a continuous function on $[a, b]$. On the basis of Theorem 2, series (7) can be integrated termwise to obtain the series

$$\int_{x_0}^{x} \varphi(t)\,dt = \int_{x_0}^{x} u_0'(t)\,dt + \int_{x_0}^{x} u_1'(t)\,dt + \ldots \quad (a\leqslant x\leqslant b),$$

which is uniformly convergent on $[a, b]$.

Applying the Newton-Leibniz theorem, we shall have

$$\int_{x_0}^{x} \varphi(t)\,dt = \sum_{0}^{\infty} [u_k(x) - u_k(x_0)]. \tag{10}$$

The series on the right of (10) with the terms equal to the functions in the brackets is uniformly convergent on $[a, b]$, the series $\sum_{0}^{\infty} u_k(x_0)$ is, by the hypothesis, convergent and, since its terms are constant, it should be regarded as uniformly convergent on $[a, b]$; but then the series $\sum_{0}^{\infty} u_k(x)$ is also uniformly convergent on $[a, b]$ as the difference of two uniformly convergent series; let us denote its sum by $S(x)$. Then equality (10) can be rewritten as

$$S(x) = S(x_0) + \int_{x_0}^{x} \varphi(t)\,dt.$$

But the function $S(x)$ has a derivative equal to $S'(x) = \varphi(x)$, and the theorem has been proved.

Example 1. The series

$$S(x) = \frac{\cos x}{1^\alpha} + \frac{\cos 2x}{2^\alpha} + \frac{\cos 3x}{3^\alpha} + \ldots \tag{11}$$

for $\alpha > 1$ is uniformly convergent throughout the real axis by Weierstrass' test, since

$$|n^{-\alpha} \cos nx| \leqslant n^{-\alpha}, \quad \forall x \in (-\infty, \infty)$$

and

$$\sum_{1}^{\infty} n^{-\alpha} < \infty \quad (\alpha > 1).$$

Let us formally differentiate series (11):

$$\varphi(x) = \frac{-\sin x}{1^{\alpha-1}} - \frac{\sin 2x}{2^{\alpha-1}} - \frac{\sin 3x}{3^{\alpha-1}} - \ldots. \tag{12}$$

This series uniformly converges on $(-\infty, \infty)$ as soon as $\alpha > 2$. But then for $\alpha > 2$

$$S'(x) = \varphi(x). \tag{13}$$

Let us consider the case $1 < \alpha \leqslant 2$. In this case Weierstrass' test is not applicable to series (12). But series (12) is uniformly convergent on the closed interval $[\varepsilon, 2\pi - \varepsilon]$ for any $\varepsilon > 0$ (see the preceding section, Example 4). Since series (11) is also convergent on this interval, on the basis of Theorem 3 we can assert that equality (13) takes place on the interval $[\varepsilon, 2\pi - \varepsilon]$, for an arbitrary small $\varepsilon > 0$, but then, obviously, on the interval $(0, 2\pi)$ as well.

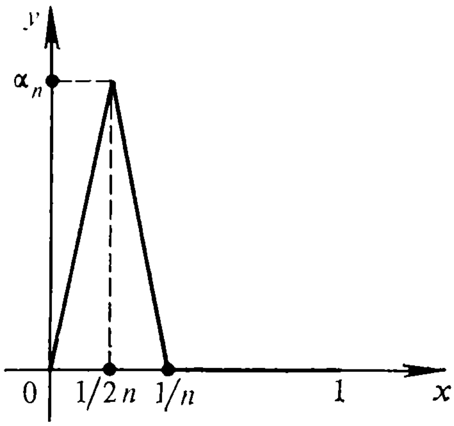

Fig. 107

Taking into account that the terms of series (11) have period 2π, we have thus proved that for the condition $1 < \alpha \leqslant 2$ it is legitimate to differentiate series (11) termwise for all $x \in (-\infty, \infty)$, excepting the points $x_k = 2k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$.

Example 2. Let the function $f_n(x)$ be continuous on $[0, 1]$, linear on each of the subintervals $[0, 1/2n]$, and $[1/2n, 1/n]$ and such that $f_n(0) = f(1/n) = 0$, $f_n(1/2n) = \alpha_n$, $f_n(x) \equiv 0$ on $[1/n, 1]$, where α_n is any sequence of numbers (Fig. 107). Then, obviously, $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in [0, 1]$, and

$$\int_0^1 f_n(x)\,dx = \int_0^{1/2n} 2n\alpha_n x\,dx + \int_{1/2n}^{1/n} 2n\alpha_n\left(\frac{1}{n} - x\right)dx = \frac{\alpha_n}{2n}.$$

Further, it is evident that

$$r_n = \sup_{0\leqslant x\leqslant 1} |f_n(x) - 0| = \alpha_n,$$

therefore the sequence $\{f_n(x)\}$ is uniformly convergent on $[0, 1]$ if and only if $\alpha_n \to 0$. The equality

$$\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx \quad (f(x) \equiv 0) \tag{14}$$

is fulfilled if and only if $\alpha_n/2n \to 0$ $(n \to \infty)$.

We see that the uniform convergence of f_n to $f = 0$ on $[0, 1]$ (i.e. when $\alpha_n \to 0$) implies the convergence of integrals (14), which agrees with Theorem 2. But the sequence $\{f_n\}$ can also be convergent *nonuniformly* and in this case property (14) may continue to hold, for instance, when $\alpha_n = 1$. This shows that the uniform convergence of a sequence is a sufficient, but not a necessary condition for a sequence of integrals to converge to the integral of the limit function. Further, for $\alpha_n = n$ not only the sequence $\{f_n\}$ converges to zero nonuniformly, but property (14) fails to hold either.

Hence, if the sequence $\{f_n\}$ is convergent nonuniformly, then two cases are possible: either the sequence of integrals $\int\limits_a^b f_n(x)\,dx$ converges to the integral of the limit function $\int\limits_a^b f(x)\,dx$, or to another number (for $\alpha_n = n$ it converges to 1/2, but not to zero), or it does not converge at all.

Example 3. The equality $(1 - z)^{-1} = 1 + z + z^2 + \ldots$ $\ldots$ $(z = \rho e^{i\theta},\ \rho < 1)$ implies that

$$\frac{1+\rho e^{i\theta}}{2(1-\rho e^{i\theta})} = \frac{1}{2} + \rho e^{i\theta} + \rho^2 e^{i2\theta} + \ldots,$$

and, separating the real and the imaginary parts, we obtain

$$P(\rho, \theta) = \frac{1}{2}\,\frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2} = \frac{1}{2} + \rho\cos\theta + \rho^2\cos 2\theta + \ldots, \tag{15}$$

$$Q(\rho, \theta) = \frac{\rho\sin\theta}{1-2\rho\cos\theta+\rho^2} = \rho\sin\theta + \rho^2\sin 2\theta + \ldots, \tag{16}$$

The function $P(\rho, \theta)$ is known as the *Poisson* kernel* and $Q(\rho, \theta)$ is *its conjugate kernel.*

These functions are harmonic functions (for $\rho < 1$), i.e. they satisfy Laplace's** differential equation in polar coordinates

$$\Delta u \equiv \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{17}$$

---

* Poisson, Simeon Denis (1781-1840). French mathematician.

** Laplace, Pierre Simon (1749-1827). French mathematician, astronomer and physicist.

Indeed, every term of series (15) is a harmonic function

$$(\rho^n \cos n\theta)'_\rho = n\rho^{n-1}\cos n\theta,\ (\rho^n \cos n\theta)''_\rho = n(n-1)\rho^{n-2}\cos n\theta,$$

$$(\rho^n \cos n\theta)''_\theta = -n^2\rho^n \cos n\theta,$$

$$\Delta(\rho^n \cos n\theta) = \rho^{n-2}\cos n\theta\,[n(n-1)+n-n^2] = 0.$$

Analogously, $\Delta(\rho^n \sin n\theta) = 0$.

The legitimacy of termwise differentiation of series (15) and (16) is stipulated by the fact that these series and the formally differentiated (once or twice) series are uniformly convergent for $\rho < 1$.

Note that the function $u(x, y)$, where x and y are Cartesian coordinates, is referred to as *harmonic in the domain* Ω of the points (x, y) if it satisfies in this domain the differential equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

In polar coordinates, this equation has the form (17).

Sec. 9.10. Multiplication of Absolutely Convergent Series

Let us consider two absolutely convergent series

$$S = \sum_{k=0}^{\infty} u_k, \quad \sigma = \sum_{l=0}^{\infty} v_l \tag{1}$$

with real or complex terms. Let us number the pairs (k, l), where $k = 0, 1, 2, \ldots$, $l = 0, 1, 2, \ldots$, in an arbitrary way

$$(k_1, l_1),\ (k_2, l_2),\ (k_3, l_3),\ \ldots . \tag{2}$$

It is important here that each indicated pair (k, l) enters into sequence (2) as its element only once, and has a certain number in this sequence. Let us prove that

$$S\sigma = \sum_{i=0}^{\infty} u_{k_i} v_{l_i}, \tag{3}$$

and the series constituting the right-hand side of (3) is absolutely convergent.

Thus, if we form series from all possible products $u_k v_l$ taken in any order, then this series is absolutely convergent and has a sum equal to $S\sigma$.

To prove this assertion, let us form series from the moduli $|u_k|$ and $|v_l|$:

$$\overline{S}=\sum_{k=0}^{\infty}|u_k|,\quad \overline{\sigma}=\sum_{l=0}^{\infty}|v_k|. \tag{1'}$$

Let us set

$$\overline{S}_N=\sum_{k=0}^{N}|u_k|,\quad \overline{\sigma}_N=\sum_{l=0}^{N}|v_l|.$$

We first order the pairs (k, l) in the following way (Fig. 108): (0, 0), (0, 1), (1, 1), (1, 0), (0, 2), (1, 2), (2, 2), (2, 1), (2, 0), . . ., then

$$\begin{aligned}\overline{S\sigma}&=\lim_{N\to\infty}\overline{S}_N\lim_{N\to\infty}\overline{\sigma}_N=\lim_{N\to\infty}(\overline{S}_N\overline{\sigma}_N)\\&=\lim_{N\to\infty}(|u_0|\cdot|v_0|+|u_0|\cdot|v_1|+|u_1|\cdot|v_1|\\&+|u_1|\cdot|v_0|+|u_0|\cdot|v_2|+|u_1|\cdot|v_2|+|u_2|\cdot|v_2|\\&+|u_2|\cdot|v_1|+|u_2|\cdot|v_0|+\ldots+|u_N|\cdot|v_0|).\end{aligned} \tag{4'}$$

This shows that the sum on the right tends, as $N\to\infty$,

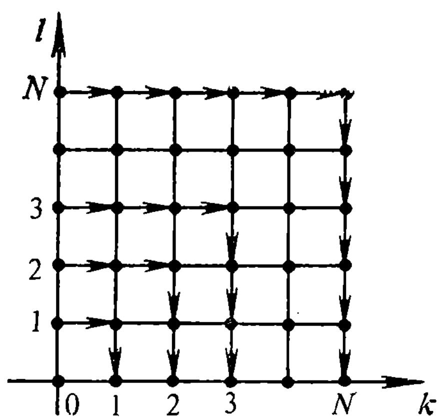

Fig. 108

to the limit equal to $\overline{S\sigma}$ and, since its terms are nonnegative, the number $\overline{S\sigma}$ is the sum of the series

$$\overline{S\sigma}=|u_0|\cdot|v_0|+|u_0|\cdot|v_1|+|u_1|\cdot|v_1|+\ldots \tag{5'}$$

Since the terms of this series are nonnegative, they can be rearranged without violating its convergence and without changing the sum $\overline{S\sigma}$.

For the time being, we have proved our assertion for series (1').

Let now

$$S_N = \sum_{k=0}^{N} u_k, \quad \sigma_N = \sum_{l=0}^{N} v_l.$$

The same as in (4'), by virtue of the convergence of series (1), we shall have

$$\begin{aligned} S\sigma &= \lim_{N\to\infty} (S_N\sigma_N) \\ &= \lim_{N\to\infty} (u_0v_0 + u_0v_1 + u_1v_1 + u_1v_0 + \ldots + u_Nv_0). \quad (4) \end{aligned}$$

Hence, there exists a limit on the right of (4), as $N \to \infty$, which is equal to $S\sigma$. But we have already proved that series (5') is convergent. This shows that the series

$$u_0v_0 + u_0v_1 + u_1v_1 + u_1v_0 + \ldots \quad (5)$$

is also absolutely convergent.

By virtue of (4), the sum of this series is equal to $S\sigma$:

$$S\sigma = u_0v_0 + u_0v_1 + u_1v_1 + u_1v_0 + \ldots .$$

We have thus proved equality (3) for one definite way of numbering the pairs (k, l). But by virtue of the absolute convergence of series (5), equality (3) will hold for any other method of numbering.

Example. The series

$$\psi(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \quad (6)$$

is absolutely convergent for any complex value of z, or in other words, it is absolutely convergent on the entire complex plane. This is easily verified by applying D'Alembert's test to a series with the general term $|z|^n/n!$.

For any two complex numbers u and v we have (the

explanations are given below)

$$\psi(u)\ \psi(v)+\left(1+u+\frac{u^2}{2!}+\ \ldots\right)\left(1+v+\frac{v^2}{2!}+\ \ldots\right)$$

$$=1+u+v+\frac{u^2}{2!}1+u\cdot v+1\,\frac{v^2}{2!}+\frac{u^3}{3!}\cdot 1$$

$$+\frac{u^2}{2!}v+u\frac{v^2}{2!}+1\cdot\frac{v^3}{3!}+\ \ldots\ =1+(u+v)$$

$$+\frac{1}{2!}(u^2+2uv+v^2)+\frac{1}{3!}(u^3+3u^2v+3uv^2+v^3)+\ \ldots$$

$$=1+(u+v)+\frac{(u+v)^2}{2!}+\frac{(u+v)^3}{3!}+\ldots=\psi(u+v). \quad (7)$$

In the second equality we arranged the products $\frac{u^k}{k!}\frac{l^l}{l!}$ in the order shown in Fig. 109 and took advantage of equality (3) for absolutely convergent series. The series thus

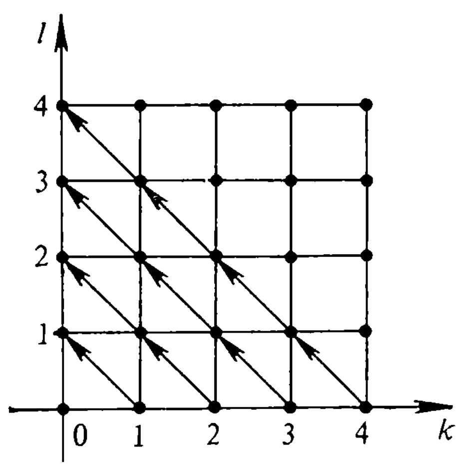

Fig. 109

obtained, as was proved in the general case, is absolutely convergent. Separate groups of terms of a convergent series may be parenthesized without violating its convergence. This is just done in the subsequent equalities.

We have thus proved the important equality

$$\psi(u+v)=\psi(u)\cdot\psi(v) \quad (8)$$

for any complex u and v. It will be discussed in more detail in Sec. 9.13.

Sec. 9.11. Power Series

A series of the form

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots, \tag{1}$$

where a_k are constant numbers and z is a variable is called a *power series*. We shall regard a_k and z as complex in the general case and only sometimes shall pass to the domain of a real variable. The letter z will, generally speaking, denote a complex variable number (a point of the complex plane), while the letter x a real variable number (a point of the real x-axis).

In the theory of power series the following basic theorem is of utmost importance.

Theorem 1 (basic). *For a power series* (1) *there exists a nonnegative number R, finite or infinite* ($0 \leqslant R \leqslant \infty$), *possessing the following properties*:

(1) *The series is absolutely convergent in an open circle in the complex plane $|z| < R$ and divergent at points z with $|z| > R$.*

(2) *The number R is determined by the formula*

$$R = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}}, \tag{2}$$

where the denominator is nothing but the upper limit (see Sec. 2.10).

Here we allow ourselves to set

$$\frac{1}{0} = \infty, \quad \frac{1}{\infty} = 0.$$

Hence, if the indicated upper limit is equal to zero, then $R = \infty$, and if it is equal to ∞, then $R = 0$.

The open circle $|z| < R$ in the complex plane is called the *circle of convergence of power series* (1). For $F = \infty$ it turns into the entire complex plane. If $R = 0$, the power series has only one point of convergence, namely, the point $z = 0$; R is called the *radius of convergence of series* (1).

Remark 1. The number R, satisfying assertion (1) of Theorem 1 is, obviously, unique.

Remark 2. If for power series (1) there exists the ordinary limit $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$, then it is equal to the upper limit $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}$. Therefore

$$R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}}.$$

The reader, who is not familiar with the notion of the upper limit, can follow the proof of Theorem 1 having assumed that the indicated limit exists for the power series under consideration. In this case, in all arguments below $\overline{\lim}$ should be replaced by lim.

The Proof of Theorem 1. Let the number R be determined by formula (2). At the point $z = 0$ the power series is convergent. We shall then hold that $|z| > 0$. Along with series (1), let us introduce a second series constructed from its moduli:

$$|a_0| + |a_1 z| + |a_2 z^2| + \dots . \tag{1'}$$

The general term of series (1′) will be denoted by

$$u_n = |a_n z^n| \quad (n = 0, 1, 2, \dots). \tag{3}$$

According to the generalized Cauchy test for the convergence of a series (see Sec. 9.4, Theorem 3, (c)),

if $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{u_n} < 1$, then series (1′) is convergent,

if $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{u_n} > 1$, then series (1′) is divergent

and the variable $\sqrt[n]{u_n}$ is unbounded, but

$$\overline{\lim} \sqrt[n]{u_n} = \overline{\lim} \sqrt[n]{|a_n z^n|} = \overline{\lim} \left(|z| \sqrt[n]{|a_n|}\right)$$
$$= |z| \overline{\lim} \sqrt[n]{|a_n|} = \frac{|z|}{R}.$$

Here we have taken the finite number $|z| > 0$ outside the sign of the upper limit.

It follows from the aforesaid: if $|z| < R$, i.e. $|z|/R < 1$, then series (1′) converges, and, together with it, absolutely converges series (1); and if $|z| > R$, i.e.

$|z|/R > 1$, then series (1′) is divergent and its general term $|a_n z^n|$ is unbounded, therefore the general term $a_n z^n$ of series (1) does not tend to zero as $n \to \infty$ and the necessary test (see Sec. 9.1) is not fulfilled for it. This shows that series (1) is divergent.

Hence, we have proved that the number R, determined from equality (2), possesses the following property:

if $|z| < R$, then series (1) is absolutely convergent, and if $|z| > R$, then series (1) is divergent.

The basic theorem has been proved.

For the sake of brevity, we shall denote the closed circle $|z| \leqslant q$ in the complex plane by σ_q.

Note that, generally speaking, a power series converges nonuniformly in the open circle $|z| < R$. However, the following theorem holds true.

Theorem 2. *Power series* (1) *is convergent absolutely and uniformly in any circle* $\sigma_q = \{z : |z| \leqslant q\}$, *where* $q < R$, *and* R *is the radius of convergence of series* (1).

Proof. Indeed, let $q < R$, then q is a real point (i.e. a point lying on the x-axis) belonging to the open circle of convergence of series (1). Therefore, at this point our power series is absolutely convergent, i.e.

$$\sum_{n=0}^{\infty} |a_n q^n| < \infty.$$

On the other hand, for $z \in \sigma_q$, we have

$$|a_n z^n| \leqslant |a_n q^n| \quad (n = 0, 1, 2, \ldots).$$

Since the right-hand members of these inequalities are independent of $z \in \sigma_q$ and the series constructed from the right-hand members is convergent, then, by Weierstrass' test (see Sec. 9.8, Theorem 1), power series (1) is convergent absolutely and uniformly in σ_q.

Theorem 3. *The sum*

$$S(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$

of a power series is a continuous function in its open circle of convergence $|z| < R$.

Indeed, the terms of our series are continuous functions of z, and the series itself is uniformly convergent in the circle σ_q, $q < R$. Consequently, by the known theorem

from the theory of uniformly convergent series (see Sec. 9.8, Theorem 2), the sum $S(z)$ of the series is a continuous function in σ_q, but then also in the entire circle $|z| < R$, since $q < R$ is arbitrary.

The radius of convergence of a power series can be computed with the aid of formula (2), but for practical purposes it is advisable to conveniently take advantage of D'Alembert's test in most cases when R is to be computed.

Let there exist the limit (finite or infinite)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|, \tag{4}$$

which will be denoted, for the time being, by $1/R_1$. Then (see (3))

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{|a_{n+1}z^{n+1}|}{|a_nz^n|}=|z|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{|z|}{R_1}$$

and, according to D'Alembert's test (Sec. 9.4, Theorem 2), if $|z| < R_1$ then series (1′), and together with it series (1), are convergent, and if $|z| > R_1$, then $|u_n| \to \infty$ and series (1) is divergent. But the number R with such properties is unique, therefore $R_1 = R$ (see Theorem 1).

Thus, we have proved that if limit (4) exists, then it is equal to $1/R$:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{R}, \tag{5}$$

where R is the radius of convergence of power series (1).

Note that we have proved in an indirect way that if limit (4) (finite or infinite) exists, then it is equal to the upper limit $\overline{\lim}\sqrt[n]{|a_n|}$.

Remark 3. We traditionally begin to study the theory of power series with Abel's theorem.

Abel's Theorem. *If power series* (1) *is convergent at a point* $z_0 \neq 0$ *of the complex plane, then it is convergent absolutely and uniformly in the closed circle* $|z| \leqslant q$, *where* q *is any number satisfying the inequalities* $0 < q < |z_0|$.

Proof. This theorem obviously follows from Theorems 1 and 2. Indeed, since z_0 is a point of convergence of series (1), $|z_0|$ cannot be greater than R. Therefore

$|z_0| \leqslant R$, $0 < q < |z_0| \leqslant R$ and $q < R$. But then, by Theorem 2, power series (1) is convergent absolutely and uniformly in the circle $|z| \leqslant q$.

Examples.

$$1 + z + z^2 + \ldots, \tag{6}$$

$$1 + \frac{z}{1^\alpha} + \frac{z^2}{2^\alpha} + \frac{z^3}{3^\alpha} + \ldots \quad (\alpha > 0), \tag{7}$$

$$1 + z + 2!z^2 + 3!z^3 + \ldots. \tag{8}$$

With the aid of formula (2) or (5) we conclude that the radius of convergence of series (6) and (7) is equal to 1, and for series (8) it is equal to zero.

The sum of series (6) (called the geometric series) is equal to $(1 - z)^{-1}$ in the open circle $|z| < 1$; its remainder is

$$r_n(z) = \sum_{n+1}^{\infty} z^k = \frac{z^{n+1}}{1-z} \to 0 \quad (n \to \infty).$$

But the convergence of the series in the indicated open circle is not uniform. For instance, this can be demonstrated by the fact that even for positive $z = x$ satisfying the condition $0 < x < 1$ there is no number N such that the inequality

$$\varepsilon > \frac{x^{n+1}}{1-x} \tag{9}$$

is satisfied for all $n > N$ and all $x \in [0, 1]$.

In fact, if x is taken arbitrarily close to 1, then the numerator of the right-hand member will also be close to 1, and the denominator will be close to zero; thus, the fraction in the right-hand side of (9) can be made greater than ε.

For $\alpha > 1$ series (7) is uniformly convergent throughout the closure of its circle of convergence, since for $|z| \leqslant 1$

$$|z^\alpha k^{-\alpha}| \leqslant k^{-\alpha} \quad \text{and} \quad \sum k^{-\alpha} < \infty.$$

If $\alpha = 1$, then at the point $z = 1$, lying on the boundary of the circle of convergence, series (7) is divergent.

Series (8) is convergent only at the point $z = 0$,

Sec. 9.12. Differentiation and Integration of Power Series

Theorem 1. *The radii of convergence of the power series*

$$a_0 + a_2 z + a_2 z^2 + \dots \tag{1}$$

and the series

$$a_1 + 2a_2 z + 3a_3 z^2 + \dots, \tag{2}$$

obtained from series (1) *by formal differentiation, coincide.*

Remark. The definition of the continuity and the derivative of a function of a complex variable $f(z)$ is the same as in the case of a function of a real variable. However, it should be borne in mind that the δ-neighbourhood of a point z_0 is an open circle of radius δ with centre at the point z_0. Proceeding from this definition, the derivative of the power function z^n is computed by the formula $(z^n)' = nz^{n-1}$.

The Proof of Theorem 1. Let R be the radius of convergence of series (1) and R' be the radius of convergence of series (2). We shall prove the theorem in the assumption that the limit

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R} \tag{3}$$

(finite or infinite) exists. We have

$$\frac{1}{R'} = \lim_{n\to\infty} \sqrt[n]{|na_n|} = \lim_{n\to\infty} \sqrt[n]{n} \cdot \lim_{n\to\infty} \sqrt[n]{|a_n|} = 1 \cdot \frac{1}{R} = \frac{1}{R},$$

consequently, $R = R'$.

In the general case, when limit (3) is not existent, there takes place

$$\varlimsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R}$$

and then

$$\frac{1}{R'} = \varlimsup_{n\to\infty} \sqrt[n]{n|a_n|} = \lim_{n\to\infty} \sqrt[n]{n}\, \varlimsup_{n\to\infty} \sqrt[n]{|a_n|} = 1 \cdot \frac{1}{R} = \frac{1}{R}.$$

But it is required to justify the second equation, i.e. we have to prove that if $\alpha_n, \beta_n > 0$ and $\alpha_n \to 1$, then

$$\varlimsup (\alpha_n \beta_n) = \varlimsup \beta_n. \tag{4}$$

Indeed, there exists a subsequence $\{n_k\}$ such that

$$\overline{\lim}\,\beta_n = \lim \beta_{n_k} = \lim \alpha_{n_k} \cdot \lim \beta_{n_k}$$
$$= \lim (\alpha_{n_k}\beta_{n_k}) \leqslant \overline{\lim}\,(\alpha_n\beta_n). \tag{5}$$

There also exists a subsequence $\{n_k\}$ such that

$$\overline{\lim}\,(\alpha_n\beta_n) = \lim (\alpha_{n_k}\beta_{n_k}) = \frac{\lim (\alpha_{n_k}\beta_{n_k})}{\lim \alpha_{n_k}} = \lim \beta_{n_k} \leqslant \overline{\lim}\,\beta_n. \tag{6}$$

From (5) and (6) follows (4).

Theorem 2. *It is legitimate to formally differentiate the power series*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots \quad (|z| < R) \tag{7}$$

within the limits of its open circle of convergence $|z| < R$, *i.e. the formula*

$$f'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \ldots, \quad |z| < R \tag{8}$$

holds true.

Proof. We shall prove this theorem only in the assumption that $z = x$ is a real variable which will enable us to reduce the problem to the well-known fact from the theory of real series.

And so, power series (7) for a real variable has the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots, \quad -R < x < R. \tag{7'}$$

Now this series has not a circle, but an interval of convergence $(-R, R)$. The corresponding formally differentiated series has the form

$$\varphi(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots. \tag{8'}$$

Its sum is designated for the time being by $\varphi(x)$. On the basis of the preceding theorem, it is convergent in the interval $(-R, R)$. Both series, as we know, are uniformly convergent in the closed interval $[-q, q]$, where $q < R$. The terms of the second series are continuous, being derivatives of the corresponding terms of the first series. But then, on the basis of Theorem 3 proved in Sec. 9.9, the equality

$$\varphi(x) = f'(x) \tag{9}$$

is fulfilled in the closed interval $[-q, q]$ and, consequently, also in the interval $(-R, R)$, since $q < R$ is arbitrary.

Note that, by virtue of the above proved theorem, series (1) may be differentiated term by term as many times as desired. At the kth stage we shall obtain the equality

$$f^{(k)}(z) = k! a_k + (k+1) k \ldots 2 a_{k+1} z + \ldots,$$

which is true for all z with $|z| < R$. Setting $z = 0$, we obtain

$$f^{(k)}(0) = k! a_k$$

or

$$a_k = \frac{f^{(k)}(0)}{k!} \quad (k = 0, 1, 2, \ldots).$$

Hence, it follows, in particular, that the *expansion of a function* $f(z)$ *into a power series* (see (1)) *in a certain circle* $|z| < R$ *(or in the interval* $(-R < x < R)$ *if a function* $f(x)$ *of a real variable* x *is under consideration) is unique.*

Thus, the sum $f(z)$ of power series (7), having a radius of convergence $R > 0$ can also be written in the following way:

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \ldots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k. \quad (10)$$

The series on the right in (10) is called *Taylor's series for the function* $f(z)$ *in powers of* z.

We have obtained that *if power series* (1) *has a radius of convergence* $R > 0$, *then it is Taylor's series of its sum* $f(z)$.

A thorough investigation of term-by-term integration of power series would involve the notion of a line integral of a complex function. Therefore we shall confine ourselves to considering the integration problem for power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots \quad (11)$$

in the real variable x $(z = x)$.

Let us take a series of type (11) convergent in the interval $(-R, R)$, where $0 < R \leqslant \infty$. The numbers a_k may be either real or complex. Choosing a point $x_0 \in (-R, R)$ which will be regarded as fixed and denoting by x the variable point belonging to $(-R, R)$, we can find $q > 0$

such that

$$-R < -q < x_0, \quad x < q < R.$$

Power series (11) is uniformly convergent in the closed interval $[-q, q]$ lying strictly inside the interval of convergence of the series. Therefore, it can be integrated termwise (see Sec. 9.9, Theorem 2) over the closed interval with end points x_0 and x:

$$\int_{x_0}^{x} f(t)\,dt = a_0(x-x_0) + \frac{a_1}{2}(x^2-x_0^2) + \frac{a_2}{3}(x^3-x_0^3) + \ldots, \tag{12}$$

$$-R < x, x_0 < R.$$

In particular, for $x_0=0$, we get

$$\int_{x_0}^{x} f(t)\,dt = a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \ldots,$$

$$-R < x < R. \tag{13}$$

Example 1. It is obvious that

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \ldots .$$

This series is convergent in the interval $(-1, 1)$ $(R = 1)$. Therefore, if $x \in (-1, 1)$, then it may be integrated termwise from zero to x (the series is uniformly convergent in any subinterval belonging to the interval of convergence):

$$\int_0^x \frac{dt}{1+t^2} = \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

$$(-1 < x < 1).$$

The obtained series is also convergent for $x = +1$ (as a Leibniz series). We can prove that it converges to $\arctan 1 = \pi/4$, i.e.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots .$$

Example 2. Taylor's series for the function e^{-t^2} has the form (see Sec. 4.16)

$$e^{-t^2}=1-t^2+\frac{t^4}{2!}-\frac{t^6}{3!}+\dots,$$

and for it $R=\infty$. Therefore this series may be integrated termwise:

$$\int_0^x e^{-t^2}\,dt=x-\frac{x^3}{3}+\frac{x^5}{5\cdot 2!}-\frac{x^7}{7\cdot 3!}+\dots,$$

i.e. we have obtained the expression of Poisson's integral in terms of a power series.

Example 3. Taylor's series for the function $y=\sin x$ has the form (see Sec. 4.16)

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots.$$

It converges over the entire axis. Hence, for $x\neq 0$, we have

$$\frac{\sin x}{x}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\dots. \qquad (14)$$

Setting $\left.\frac{\sin x}{x}\right|_{x=0}=1$, we conclude that equality (14) is valid for $x=0$ as well. Series (14) is uniformly convergent in any finite interval of the real axis. Integrating this power series, we get

$$\int_0^x \frac{\sin t}{t}\,dt=x-\frac{x^3}{3\cdot 3!}+\frac{x^5}{5\cdot 5!}-\frac{x^7}{7\cdot 7!}+\dots.$$

Example 4. Taylor's series for the function $y=\cos x^2$ has the form (see Sec. 4.16)

$$\cos x^2=1-\frac{x^4}{2!}+\frac{x^8}{4!}-\frac{x^{12}}{6!}+\dots.$$

It is convergent in $(-\infty, \infty)$. Integrating this power series, we obtain the Fresnel cosine integral

$$\int_0^x \cos t^2\,dt=x-\frac{x^5}{5\cdot 2!}+\frac{x^9}{9\cdot 4!}-\frac{x^{13}}{13\cdot 6!}+\dots.$$

Example 5. Since

$$(\sinh x)^{(n)} = \begin{cases} \sinh x, & \text{if} \quad n = 2k, \\ \cosh x, & \text{if} \quad n = 2k+1, \end{cases}$$

then

$$(\sinh x)^{(n)}|_{x=0} = \begin{cases} 0, & \text{if} \quad n = 2k, \\ 1, & \text{if} \quad n = 2k+1, \end{cases}$$

therefore Taylor's series for the function $\sinh x$ will be written as follows:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots . \tag{15}$$

Since this power series is convergent throughout the real axis (apply D'Alembert's test!), we are allowed to differentiate it termwise:

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \tag{16}$$

(the series on the right in (16) is uniformly convergent in any finite interval).

Sec. 9.13. The Functions e^z, $\sin z$, and $\cos z$ of Complex Variable z

The functions e^x, $\sin x$, and $\cos x$ of the real variable x are well known. They are defined throughout the real axis ($-\infty < x < \infty$).

We know from Sec. 4.16 that these functions are expanded into power series:

$$\left.\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots . \end{aligned}\right\} \tag{1}$$

These are Taylor's series for the functions in powers of x.

In Sec. 5.3 the function e^{ix}, where x is a real variable, was defined by means of Euler's formula

$$e^{ix} = \cos x + i \sin x. \qquad (2)$$

Replacing in (2) $\cos x$ and $\sin x$ by their power series, we obtain the expansion of e^{ix} in powers of x:

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$
$$= 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \dots.$$

The function e^z for any complex $z = x + iy$ is naturally defined in the following way:

$$e^z = e^{x+iy} = e^x e^{iy}.$$

Hence,

$$e^z = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots\right)\left(1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \dots\right)$$
$$= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots.$$

In the second equality we took advantage of the property (multiplication of absolutely convergent series) which had been derived before (see Sec. 9.10, (7)).

We have thus obtained that the function e^z of the complex variable z is expanded into a power series in powers of z

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots, \qquad (3)$$

convergent to it throughout the complex plane.

Series (3) is Taylor's series for the function e^z in powers of z.

The radius of convergence of series (3) $R = \infty$ and from the general properties of power series (see Sec. 9.10) it follows that series (3) is absolutely convergent for any complex z; it uniformly converges (to e^z) in the circle $|z| \leqslant q$ for any arbitrarily large positive number q.

The functions $\cos z$ and $\sin z$ of the complex variable z are naturally defined as sums of the following power se-

ries:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots,$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots.$$

Both of these series have the radius of convergence $R = \infty$ and, thus, both functions are defined for any complex z.

It is readily checked by comparing the respective power series that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{4}$$

for any complex z.

Now, making use of the properties of the exponential function e^u (of a complex u), we easily obtain the following formulas

$$\cos(u + v) = \cos u \cdot \cos v - \sin u \cdot \sin v,$$

$$\sin(u + v) = \sin u \cdot \cos v + \cos u \cdot \sin v,$$

which are true for any complex u and v.

These formulas are thus the generalization of the well-known trigonometric formulas, where u and v were regarded as real variables. Note that in the complex plane the functions $\sin z$ and $\cos z$ possess not all the properties of the ordinary functions $\sin x$ and $\cos x$. In particular, they are unbounded in the complex plane.

By virtue of (4), for a real x we have

$$\cos ix = \frac{e^{-x} + e^x}{2} = \cosh x \to \infty, \quad x \to \infty, \tag{5}$$

$$\sin ix = \frac{e^{-x} - e^x}{2i} = i \sinh x \to \infty, \quad x \to \infty. \tag{6}$$

By the way, formulas (5) and (6) establish the relationship between "complex trigonometry" and "hyperbolic trigonometry".

The function $z = \ln w$ of the complex variable w is defined as the inverse of the function

$$w = e^z. \tag{7}$$

If we write $w \neq 0$ in the exponential form

$$w = \rho e^{i\theta} \; (\rho = |w| > 0),$$

equality (7) is then written as

$$\rho e^{i\theta} = e^x e^{iy} \quad (z = x + iy),$$

whence

$$\rho = e^x, \quad \theta = y - 2k\pi,$$

i.e.

$$x = \ln \rho, \; y = \theta + 2k\pi \quad (k = 0, \pm 1, \pm 2, \ldots).$$

Therefore,

$$\begin{aligned} z = \ln w = x + iy &= \ln \rho + i\,(\theta + 2k\pi) \\ &= \ln |w| + i \operatorname{Arg} w = \ln |w| + i \arg w + i2k\pi \\ &\qquad (k = 0, \pm 1, \pm 2, \ldots), \end{aligned} \tag{8}$$

where $\ln |w|$ ($|w| > 0$) is understood in the ordinary sense. It is seen from (8) that $\ln w$ ($w \neq 0$), together with $\operatorname{Arg} w$, are many-valued functions of w, irrespective of whether w is real or complex.

For instance, from the point of view of the theory of functions of a complex variable the natural logarithm of 1 ($\ln 1$) is equal to any of the numbers $2k\pi i$ ($k = 0, \pm 1, \pm 2, \ldots$). In real analysis $\ln 1$ is understood as the number 0 which is one of the above values.

Here we shall not consider further topics of the theory of functions of a complex variable and shall only confine ourselves to the following remark concerning the formula

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \quad (-1 < x \leqslant 1),$$

which was derived in Sec. 4.16 for real x's. If x is replaced in the right-hand side of the above formula by a complex number z with $|z| < 1$, then the series continues to be convergent. We can say that its sum is equal to $\ln(1 + z)$, where the logarithm is understood in the sense of the above general definition, more precisely, its sum is equal to one of the branches of the infinite-valued function $\ln(1 + z)$.

Functions of a complex variable which can be expanded into convergent power series (Taylor's series) are called *analytic functions*. They are studied in a special division of higher mathematics called the theory of analytic functions or the theory of functions of a complex variable*.

Finally, we should like to note that if in the power series in powers of u

$$a_0 + a_1 u + a_2 u^2 + \ldots$$

with the circle of convergence $|u| < R$ we set $u = z - z_0$, where z_0 is a fixed number (generally speaking, a complex one), then we shall obtain the series

$$a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \ldots,$$

called a *power series in powers of* $z - z_0$.

It is convergent in the circle (of convergence) $|z - z_0| < R$ and divergent for z's satisfying the inequality $|z - z_0| > R$.

Sec. 9.14. Series in Approximate Computations

This section deals with approximate computation of values of elementary functions.

The polynomial

$$P_n(x) = a_0 + a_1 x + \ldots + a_n x^n$$

is the simplest elementary function.

The computation of this function for $x = x_0$ is reduced to carrying out a finite number of operations of addition and multiplication. The value of this function at the point x_0 can be readily found with any degree of accuracy. With the aid of an electronic computer, this can be done rather quickly.

Other elementary functions, such as $\sin x$, $\arctan x$, ..., as was shown above, are expanded into Taylor's series in powers of x.

The error introduced by replacing a function (the sum of a series) by Taylor's polynomial can be determined by estimating the remainder of the series.

* See our book *Differential Equations. Series. Multiple Integrals. Functions of a Complex Variable* which completes the series "Higher Mathematics".

Let us consider the power series

$$f(x) = c_0 + c_1x + c_2x^2 + \dots \quad (-R < x < R) \qquad (1)$$

with the interval of convergence $(-R, R)$. Strictly inside the interval of convergence, it converges to $f(x)$ at the rate of a decreasing geometric progression.

Indeed, let q_1 and q be arbitrary numbers satisfying the inequalities $0 < q_1 < q < R$. Then series (1) is convergent at the point $x = q$ and its terms form a bounded sequence $(|c_nq^n| \leqslant M, \ \forall n)$. Therefore for all $x \in [-q_1, q_1]$

$$|c_nx^n| = \left|c_nq^n\left(\frac{x}{q}\right)^n\right| \leqslant M\left(\frac{q_1}{q}\right)^n,$$

where $q_1/q < 1$.

We see that it is advantageous to utilize a power series to compute the values of a function $f(x)$ at points lying strictly inside the interval of convergence.

If x is an end point of the interval $(-R, R)$, then at this point if the series is convergent, then it converges slower than a decreasing geometric progression. It does so slow that we usually find it not advantageous to directly use power series (1) for computing the value of f at the indicated end point. Below we shall illustrate these facts by considering particular examples.

We shall begin our computations with the number π. It was shown in Sec. 9.12 (Example 1) that

$$\operatorname{arc\,tan} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \quad (-1 < x < 1). \qquad (2)$$

At the point $x = 1$ this series is convergent as well. Let us prove that it converges just to $\operatorname{arc\,tan} 1 = \pi/4$. In Sec. 9.12 this fact was not proved. Consider the identity

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \dots + (-1)^n x^{2n} + (-1)^{n+1}\frac{x^{2n+2}}{1+x^2}.$$

Integrating this identity over $[0, 1]$, we have

$$\int_0^1 \frac{dx}{1+x^2} = \operatorname{arc\,tan} 1 = \frac{\pi}{4} = \int_0^1 dx - \int_0^1 x^2\,dx$$

$$+\dots+(-1)^n \int_0^1 x^{2n}\,dx+(-1)^{n+1}\int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx$$
$$=1-\frac{1}{3}+\frac{1}{5}-\dots+(-1)^n\frac{1}{2n+1}+\alpha_n,$$

where

$$\alpha_n=(-1)^{n+1}\int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx.$$

It is easy to see that

$$|\alpha_n|\leqslant\int_0^1 x^{2n+2}\,dx=\frac{1}{2n+3}\to 0,\quad n\to\infty.$$

Hence it follows that

$$\left|\operatorname{arc\,tan} 1-\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\right|\to 0,\quad n\to\infty,$$

i.e. arc tan 1 is the sum of the series

$$\operatorname{arc\,tan} 1=\frac{\pi}{4}=\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$

or

$$\pi=4\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}. \tag{3}$$

We see that this series converges slower than any decreasing geometric progression.

To compute the number π with the aid of series (3) to within 10^{-6}, it is necessary to take so many terms of series (3) that the remainder should be less than 10^{-6}. Since (3) is a Leibniz series, its remainder is less than the modulus of its first term

$$|R_n|=\left|4\sum_{k=n+1}^{\infty}\frac{(-1)^k}{2k+1}\right|<\frac{4}{2n+3}.$$

Hence it is seen that for $n = 2\cdot 10^6$, $|R_n| < 10^{-6}$. Thus, it is necessary to take two million terms of series (3) to guarantee the desired accuracy of computation of the number π.

It is senseless to carry out this work manually. It can be accomplished by making use of a computer, but even a computer will not prove efficient if series (3) is used.

We are going to indicate a series converging to the number π more rapidly. To this end, let us consider the number α such that

$$\tan \alpha = 1/5.$$

Then

$$\tan 2\alpha = \frac{2 \tan \alpha}{1-\tan^2 \alpha} = \frac{2/5}{1-1/25} = \frac{5}{12},$$

$$\tan 4\alpha = \frac{2 \tan 2\alpha}{1-\tan^2 2\alpha} = \frac{120}{119},$$

$$\tan\left(4\alpha - \frac{\pi}{4}\right) = \frac{\tan 4\alpha - \tan(\pi/4)}{1+\tan 4\alpha \cdot \tan(\pi/4)} = \frac{1}{239}.$$

Hence

$$4\alpha - \frac{\pi}{4} = \operatorname{arc\,tan}(1/239),$$

$\pi = 16\alpha - 4\operatorname{arc\,tan}(1/239) = 16\operatorname{arc\,tan}(1/5) - 4\operatorname{arc\,tan}(1/239)$.

Using now series (2), we get

$$\pi = 16 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\,5^{2k+1}} - 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\,239^{2k+1}}.$$

The last two series converge rather quickly (more rapidly than a decreasing geometric progression).

We can easily check that the remainder of the first series is less than 10^{-6} beginning with $n = 4$. Therefore, as a result of computation of the four terms of the first series and two terms of the second series (to the seventh decimal place), we get

$$\pi \approx 3.141592,$$

the first five decimal digits being exact.

Computation of Logarithms

Taylor's series for the function $y = \ln(1 + x)$ can be obtained by integrating the identity

$$\frac{1}{1+x}=1-x+x^2-x^3+\dots \quad (|x|<1),$$

$$\ln(1+x)=\int_0^x \frac{dx}{1+x}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots . \quad (4)$$

For $x = 1$ the given series converges to $\ln 2$. Indeed, integrating the identity

$$\frac{1}{1+x}=1-x+x^2-\dots+(-1)^n x^n+(-1)^{n+1}\frac{x^{n+1}}{1+x}$$

over the closed interval $[0, 1]$, we obtain

$$\ln 2=1-\frac{1}{2}+\frac{1}{3}-\dots+(-1)^n\frac{1}{n+1}+\beta_n,$$

where

$$|\beta_n|=\left|\int_0^1 \frac{x^{n+1}}{1+x}\,dx\right|\leqslant\int_0^1 x^{n+1}\,dx=\frac{1}{n+2}\to 0,\ n\to\infty.$$

For $x = 1$ series (4), the same as series (3), converges slowly.

Replacing x by $-x$ in (4), we obtain

$$\ln(1-x)=-\sum_{k=1}^{\infty}\frac{x^k}{k}. \quad (5)$$

Subtracting (5) from (4), we have

$$\ln\frac{1+x}{1-x}=2\sum_{k=1}^{\infty}\frac{x^{2k+1}}{2k+1}. \quad (6)$$

This equality is just used for computing the logarithms of natural numbers. For instance, setting $x = 1/3$, we obtain

$$\ln 2=2\sum_{k=1}^{\infty}\frac{1}{(2k+1)\,3^{2k+1}}, \quad (7)$$

where the series on the right converges even faster than the geometric progression.

To compute ln 2 with an accuracy to 10^{-5}, it is sufficient to take five terms of series (7):

$$\ln 2 \approx 0.693146$$

(each term being computed with six digits after the decimal point).

In general, putting $x = \frac{1}{2m+1}$, where m is a natural number, we obtain

$$\frac{1+x}{1-x} = \frac{m+1}{m},$$

$$\ln(m+1) = \ln m + 2\sum_{k=1}^{\infty} \frac{1}{(2k+1)(2m+1)^{2k+1}}. \quad (8)$$

Setting in succession $m = 2, 3, \ldots$, we find ln 3, ln 4, ... The series in the right-hand side of (8) converges very rapidly.

Computation of Roots

In Sec. 4.16 we obtained Taylor's series for the function $f(x) = (1 + x)^{\alpha}$:

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} x^k, \quad (9)$$

which is called a *binomial series*. It is known that for $x = \pm 1$ series (9) not always converges, and if it does, then in a slow manner. Therefore, if, for instance, it is required to compute $\sqrt{2}$, then it is not reasonable to take advantage of formula (9) for $x = 1$, $\alpha = 1/2$. It is advisable to proceed in the following way: the radicand should be transformed so as to differ from unity but little:

$$\sqrt{2} = \sqrt{\frac{2 \cdot 25 \cdot 49}{25 \cdot 49}} = \frac{7}{5}\sqrt{\frac{50}{49}} = \frac{7}{5}\left(1 + \frac{1}{49}\right)^{1/2},$$

or

$$\sqrt{2} = \frac{7}{5}\frac{1}{\sqrt{49/50}} = \frac{7}{5}\left(1 - \frac{1}{50}\right)^{-1/2}$$

$$= \frac{7}{5}\left(1 - \frac{2}{100}\right)^{-1/2}. \quad (10)$$

The numbers 25 and 49 are found as follows: we write the squares of natural numbers

$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ \ldots, \tag{11}$$

and then write out a row of numbers obtained from (11) by multiplying each of its numbers by the radicand (in this case by 2):

$$2,\ 8,\ 18,\ 32,\ 50,\ 72,\ 98,\ 128,\ \ldots\ . \tag{12}$$

In rows (11) and (12) we look for the required numbers so that their ratio would be close to unity. Among the above written numbers, the sought-for numbers are 49 and $50 = 2 \times 25$.

Extending these rows, we can find another pair of matched numbers: 289 and 288, i.e.

$$\sqrt{2}=\sqrt{\frac{2\cdot 144\cdot 289}{144\cdot 289}}=\frac{17}{12}\sqrt{\frac{288}{289}}$$

$$=\frac{17}{12}\left(1+\frac{1}{288}\right)^{-1/2}. \tag{13}$$

Now we may use series (9). For instance, by virtue of (13), for $x = 1/288$ we obtain

$$\sqrt{2}=\frac{17}{12}\sum_{k=0}^{\infty}$$

$$\times\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots\left(-\frac{1}{2}-k+1\right)}{k!}\,\frac{1}{288^k}. \tag{14}$$

The series in the right-hand side of (14) converges very rapidly. Besides, this is an alternating series, i.e. its remainder is less than the modulus of the first term of this remainder.

Let us write series (14) in the expanded form:

$$\sqrt{2}=\frac{17}{12}\left\{1-\frac{1}{2\cdot 288}+\frac{1\cdot 3}{2^2\cdot 2!\,288^2}-\frac{1\cdot 3\cdot 5}{2^3\cdot 3!\,288^3}+\ldots\right\}. \tag{15}$$

The third term of series (15) is less than $8 \cdot 10^{-6} < 10^{-5}$, therefore

$$\sqrt{2} \approx \frac{17}{12}\left(1 - \frac{1}{576}\right) = 1.414207 \ldots$$

with four correct digits.

Note that, proceeding from (10), $\sqrt{2}$ is computed in a very convenient way, since we directly obtain powers of 10 in the denominator. If we take the first three terms of this series, then $\sqrt{2} \approx 1.41421$.

Example. Compute $\sqrt[3]{5}$ to within 0.01.

We arrange a row of the cubes of natural numbers

$$1,\ 8,\ 27,\ 64,\ \mathbf{125},\ 216,\ \ldots$$

and the row of these numbers multiplied by 5

$$5,\ 40,\ \mathbf{135},\ 320,\ 625,\ 1080,\ \ldots .$$

Hence,

$$\sqrt[3]{5} = \sqrt[3]{\frac{5 \cdot 27 \cdot 125}{27 \cdot 125}} = \frac{5}{3}\left(1 + \frac{10}{125}\right)^{1/3}$$
$$= \frac{5}{3}\left(1 + \frac{8}{100}\right)^{1/3} = \frac{5}{3}\left\{1 + \frac{8}{3 \cdot 10^2} - \frac{2 \cdot 8^2}{3^2 \cdot 2!\, 10^4} + \frac{2 \cdot 5 \cdot 8^3}{3^3 \cdot 3!\, 10^6} - \ldots\right\},$$

the third term of the series

$$\frac{5 \cdot 8^2}{3^3 \cdot 10^4} < 0.01,$$

therefore

$$\sqrt[3]{5} \approx \frac{5}{3}\left(1 + \frac{8}{300}\right) = 1.71 \ldots$$

to within 0.01.

SUBJECT INDEX

Printed in the Union of Soviet Socialist Republics